Math 311 **Spring 2023**
Homework 5 **Drew Armstrong**

The *Laplace transform* $F(s)$ of a function $f(t)$ is defined as follows:

$$F(s) = \mathscr{L}[f(t)] = \int_0^\infty e^{-st}\,dt.$$

One can use this definition to derive the following general rules:

(1) $\mathscr{L}[t \cdot f(t)] = -F'(s)$
(2) $\mathscr{L}[e^{at} \cdot f(t)] = F(s-a)$
(3) $\mathscr{L}[f'(t)] = sF(s) - f(0)$
(4) $\mathscr{L}[f''(t)] = s^2F(s) - sf(0) - f'(0)$
(5) $\mathscr{L}[H(t-a)\cdot f(t-a)] = e^{-as}F(s)$, where $H(t)$ is the *Heaviside step function*:

$$H(t) = \begin{cases} 0 & t < 0, \\ 1 & t > 1. \end{cases}$$

Here are the transforms of some basic functions:

- $\mathscr{L}[0] = 0$
- $\mathscr{L}[1] = 1/s$
- $\mathscr{L}[e^{at}] = 1/(s-a)$
- $\mathscr{L}[t] = 1/s^2$
- $\mathscr{L}[t^n] = n!/s^{n+1}$
- $\mathscr{L}[\cos(kt)] = s/(s^2+k^2)$
- $\mathscr{L}[\sin(kt)] = k/(s^2+k^2)$

The *Dirac delta function* $\delta(t)$ satisfies $\mathscr{L}[\delta(t-a)] = e^{-as}$.

1. Using the Rules.

(a) Use rule (1) to compute

$$\mathscr{L}\left[t\cdot\sin(kt)\right] \quad \text{and} \quad \mathscr{L}\left[t\cdot\cos(kt)\right].$$

(b) Use rule (2) to compute

$$\mathscr{L}^{-1}\left[\frac{1}{(s-1)^2}\right] \quad \text{and} \quad \mathscr{L}^{-1}\left[\frac{2}{(s-3)^2+4}\right].$$

(c) Use rule (5) to compute

$$\mathscr{L}^{-1}\left[\frac{e^{-s}}{s}\right] \quad \text{and} \quad \mathscr{L}^{-1}\left[\frac{e^{-2s}}{s^2+1}\right].$$

(a): We have

$$\begin{aligned}
\mathscr{L}\left[t\cdot\sin(kt)\right] &= -\frac{d}{ds}\mathscr{L}\left[\sin(kt)\right] \\
&= -\frac{d}{ds}\,\frac{k}{s^2+k^2} \\
&= -\frac{d}{ds}k(s^2+k^2)^{-1} \\
&= -k(-1)(s^2+k^2)^{-2}(2s)
\end{aligned}$$

$$= \frac{2ks}{(s^2+k^2)^2}$$

and

$$\begin{aligned}
\mathscr{L}\left[t\cdot\cos(kt)\right] &= -\frac{d}{ds}\mathscr{L}\left[\cos(kt)\right] \\
&= -\frac{d}{ds}\frac{s}{s^2+k^2} \\
&= -\frac{(s^2+k^2)(1)-s(2s)}{(s^2+k^2)^2} \\
&= \frac{s^2-k^2}{(s^2+k^2)^2}.
\end{aligned}$$

(b): We note that $1/(s-1)^2$ is a shifted form of $1/s^2$, hence

$$\mathscr{L}^{-1}[1/(s-1)^2] = e^t\cdot\mathscr{L}^{-1}[1/s^2] = e^t\cdot t.$$

We note that $\frac{2}{(s-3)^2+4}$ is a shifted form of $\frac{2}{s^2+4}$. Hence

$$\mathscr{L}^{-1}\left[\frac{2}{(s-3)^2+4}\right] = e^{3t}\cdot\mathscr{L}^{-1}\left[\frac{2}{s^2+4}\right] = e^{3t}\cdot\sin(2t).$$

(c): Since $\mathscr{L}^{-1}[1/s]=1$ we have

$$\mathscr{L}^{-1}\left[\frac{e^{-s}}{s}\right] = H(t-1)\cdot 1 = \begin{cases}0 & t<1,\\ 1 & t>1.\end{cases}$$

Since $\mathscr{L}^{-1}[1/(s^2+1)]=\sin t$ we have

$$\mathscr{L}^{-1}\left[\frac{e^{-2s}}{s^2+1}\right] = H(t-2)\cdot\sin(t-2) = \begin{cases}0 & t<2,\\ \sin(t-2) & t>2.\end{cases}$$

2. Some Small Problems. Solve using Laplace transforms:

(a) $x'(t)=x(t)$; $x(0)=1$
(b) $x'(t)=x(t)+1$; $x(0)=1$
(c) $x'(t)=x(t)+e^t$; $x(0)=1$

(a): Applying Laplace transforms gives

$$\begin{aligned}
x'(t) &= x(t) \\
sX - x(0) &= X \\
sX - 1 &= X && x(0)=1 \\
sX - X &= 1 \\
(s-1)X &= 1 \\
X &= 1/(s-1) \\
x(t) &= \mathscr{L}^{-1}[1/(s-1)] \\
&= e^t.
\end{aligned}$$

This answer is not a surprise. We are just solving it with a new method.

(b): Applying Laplace transforms gives

$$\begin{aligned} x'(t) &= x(t) + 1 \\ sX - 1 &= X + 1/s \\ sX - X &= 1 + 1/s \\ (s-1)X &= 1 + 1/s \\ X &= \frac{1}{s-1} + \frac{1}{s(s-1)} \\ x(t) &= \mathscr{L}^{-1}\left[\frac{1}{s-1}\right] + \mathscr{L}^{-1}\left[\frac{1}{s(s-1)}\right] \\ x(t) &= e^t + \mathscr{L}^{-1}\left[\frac{1}{s(s-1)}\right]. \end{aligned}$$

Now we use partial fractions:

$$\begin{aligned} \frac{1}{s(s-1)} &= \frac{A}{s} + \frac{B}{s-1} \\ \frac{1}{s(s-1)} &= \frac{A(s-1) + Bs}{s(s-1)} \\ 1 &= A(s-1) + Bs. \end{aligned}$$

Substituting $s = 0$ and $s = 1$ gives $A = -1$ and $B = 1$, hence

$$\begin{aligned} x(t) &= e^t + \mathscr{L}^{-1}\left[\frac{-1}{s} + \frac{1}{s-1}\right] \\ &= e^t - \mathscr{L}^{-1}\left[\frac{1}{s}\right] + \mathscr{L}^{-1}\left[\frac{1}{s-1}\right] \\ &= e^t - 1 + e^t \\ &= 2e^t - 1. \end{aligned}$$

Again, this is not a surprise.

(c): Applying Laplace transforms gives

$$\begin{aligned} x'(t) &= x(t) + t \\ sX - 1 &= X + 1/s^2 \\ sX - X &= 1 + 1/s^2 \\ (s-1)X &= 1 + 1/s^2 \\ X &= \frac{1}{s-1} + \frac{1}{s^2(s-1)} \\ x(t) &= \mathscr{L}^{-1}\left[\frac{1}{s-1}\right] + \mathscr{L}^{-1}\left[\frac{1}{s^2(s-1)}\right] \\ x(t) &= e^t + \mathscr{L}^{-1}\left[\frac{1}{s^2(s-1)}\right]. \end{aligned}$$

Now we use partial fractions:

$$\frac{1}{s^2(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1}$$

$$\frac{1}{s^2(s-1)} = \frac{As(s-1)+B(s-1)+Cs^2}{s^2(s-1)}$$
$$1 = As(s-1)+B(s-1)+Cs^2.$$

We can substitute three values of s to get three equations for A, B, C or we can expand and compare coefficients:

$$0s^2 + 0s + 1 = (A+C)s^2 + (B-A)s - B.$$

The equation $1 = -B$ gives $B = -1$. Then the equation $B - A = 0$ gives $A = -1$. Then the equation $A + C = 0$ gives $C = 1$. We conclude that

$$\begin{aligned} x(t) &= e^t + \mathscr{L}^{-1}\left[\frac{1}{s^2(s-1)}\right] \\ &= e^t + \mathscr{L}^{-1}\left[\frac{-1}{s} + \frac{-1}{s^2} + \frac{1}{s-1}\right] \\ &= e^t - \mathscr{L}^{-1}\left[\frac{1}{s}\right] - \mathscr{L}^{-1}\left[\frac{1}{s^2}\right] + \mathscr{L}^{-1}\left[\frac{1}{s-1}\right] \\ &= e^t - 1 - t + e^t \\ &= 2e^t - 1 - t. \end{aligned}$$

Previously we solved this equation using integrating factors and the method of undetermined coefficients. Undetermined coefficients is the fastest method, but it requires a good guess. Integrating factors involves integration by parts. Laplace transforms involves partial fractions. Which method is best?

3. A Bigger Problem.

(a) Find the partial fraction expansion of $\frac{1}{(s-2)(s-3)}$.
(b) Find the partial fraction expansion of $\frac{s}{(s-2)(s-3)}$.
(c) Find the partial fraction expansion of $\frac{1}{s(s-2)(s-3)}$.
(d) Use Laplace transforms together with (a), (b), (c) to solve the initial value problem:

$$x''(t) - 5x'(t) + 6x(t) = 1; \quad x(0) = 5, \quad x'(0) = 7.$$

(a): We are looking for A, B such that

$$\frac{1}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3}$$
$$\frac{1}{(s-2)(s-3)} = \frac{A(s-3)+B(s-2)}{(s-2)(s-3)}$$
$$1 = A(s-3)+B(s-2).$$

Putting $s = 2$ and $s = 3$ gives $A = -1$ and $B = 1$, hence

$$\frac{1}{(s-2)(s-3)} = \frac{-1}{s-2} + \frac{1}{s-3}.$$

(b): We are looking for A, B such that

$$\frac{s}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3}$$
$$\frac{s}{(s-2)(s-3)} = \frac{A(s-3)+B(s-2)}{(s-2)(s-3)}$$

$$s = A(s-3) + B(s-2).$$

Putting $s = 2$ and $s = 3$ gives $A = -2$ and $B = 3$, hence

$$\frac{s}{(s-2)(s-3)} = \frac{-2}{s-2} + \frac{3}{s-3}.$$

(c): We are looking for A, B, C such that

$$\frac{1}{s(s-2)(s-3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s-3}$$
$$\frac{1}{s(s-2)(s-3)} = \frac{A(s-2)(s-3) + Bs(s-3) + Cs(s-2)}{s(s-2)(s-3)}$$
$$1 = A(s-2)(s-3) + Bs(s-3) + Cs(s-2).$$

Putting $s = 0$, $s = 2$ and $s = 3$ gives $A = 1/6$, $B = -1/2$ and $C = 1/3$, hence

$$\frac{1}{s(s-2)(s-3)} = \frac{1/6}{s} + \frac{-1/2}{s-2} + \frac{1/3}{s-3}.$$

(d): Apply Laplace transforms to get

$$x''(t) - 5x'(t) + 6x(t) = 1$$
$$(s^2X - sx(0) - x'(0)) - 5(sX - x(0)) + 6X = 1/s$$
$$(s^2X - 5s - 7) - 5(sX - 5) + 6X = 1/s \qquad x(0) = 5 \text{ and } x'(0) = 7$$
$$s^2 - 5s - 7 - 5sX + 25 + 6X = 1/s$$
$$(s^2 - 5s + 6)X = -18 + 5s + 1/s$$
$$(s-2)(s-3)X = -18 + 5s + 1/s.$$

Dividing both sides by $(s-2)(s-3)$ gives

$$(*) \qquad X = -18 \cdot \frac{1}{(s-2)(s-3)} + 5 \cdot \frac{s}{(s-2)(s-3)} + \frac{1}{s(s-2)(s-3)}.$$

From parts (a), (b), (c) we have

$$\mathscr{L}^{-1}\left[\frac{1}{(s-2)(s-3)}\right] = \mathscr{L}^{-1}\left[\frac{-1}{s-2} + \frac{1}{s-3}\right] = -e^{2t} + e^{3t}$$

and

$$\mathscr{L}^{-1}\left[\frac{s}{(s-2)(s-3)}\right] = \mathscr{L}^{-1}\left[\frac{-2}{s-2} + \frac{3}{s-3}\right] = -2e^{2t} + 3e^{3t}$$

and

$$\mathscr{L}^{-1}\left[\frac{1}{s(s-2)(s-3)}\right] = \mathscr{L}^{-1}\left[\frac{1/6}{s} + \frac{-1/2}{s-2} + \frac{1/3}{s-3}\right] = \frac{1}{6} - \frac{1}{2}e^{2t} + \frac{1}{3}e^{3t}.$$

Thus, applying $\mathscr{L}^{-1}$ to $(*)$ gives

$$x(t) = -18(-e^{2t} + e^{3t}) + 5(-2e^{2t} + 3e^{3t}) + \left(\frac{1}{6} - \frac{1}{2}e^{2t} + \frac{1}{3}e^{3t}\right)$$

$$= \frac{1}{6} + \frac{15}{2}e^{2t} - \frac{8}{3}e^{3t}.$$

That was a lot of computation. The method of eigenvalues would have been faster.

4. Resonance. Consider the following initial value problem:

$$x''(t) + 4x(t) = \cos(\omega t); \quad x(0) = x'(0) = 0.$$

(a) First suppose that $\omega \neq 2$. In this case solve for A, B, C, D in the expansion:

$$\frac{s}{(s^2+4)(s^2+\omega^2)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+\omega^2}.$$

(b) Use part (a) and Laplace transforms to solve the initial value problem when $\omega \neq 2$.
(c) Use Problem 1(a) and Laplace transforms to solve the initial value problem when $\omega = 2$.

(a): Assume that $\omega \neq 2$. We are looking for A, B, C, D so that

$$\begin{aligned}
\frac{s}{(s^2+4)(s^2+\omega^2)} &= \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+\omega^2} \\
\frac{s}{(s^2+4)(s^2+\omega^2)} &= \frac{(As+B)(s^2+\omega^2) + (Cs+D)(s^2+4)}{(s^2+4)(s^2+\omega^2)} \\
s &= (As+B)(s^2+\omega^2) + (Cs+D)(s^2+4) \\
0s^3 + 0s^2 + 1s + 0 &= (A+C)s^3 + (B+D)s^2 + (A\omega^2+4C)s + (B\omega^2+4D).
\end{aligned}$$

Comparing coefficients gives four equations:

$$\begin{cases} A + C &= 0, \\ B + D &= 0, \\ A\omega^2 + 4C &= 1, \\ B\omega^2 + 4D &= 0. \end{cases}$$

The first two equations give $C = -A$ and $D = -B$. Since $\omega \neq 2$ we have $\omega^2 - 4 \neq 0$,[1] hence the fourth equation gives

$$\begin{aligned}
B\omega^2 + 4D &= 0 \\
B\omega^2 - 4B &= 0 \\
(\omega^2 - 4)B &= 0 \\
B &= 0.
\end{aligned}$$

Finally, the third equation gives

$$\begin{aligned}
A\omega^2 + 4C &= 1 \\
A\omega^2 - 4A &= 1 \\
(\omega^2 - 4)A &= 1 \\
A &= 1/(\omega^2 - 4).
\end{aligned}$$

We conclude that

$$\begin{aligned}
\frac{s}{(s^2+4)(s^2+\omega^2)} &= \frac{s/(\omega^2-4)}{s^2+4} + \frac{-s/(\omega^2-4)}{s^2+\omega^2} \\
&= \frac{1}{\omega^2-4}\left(\frac{s}{s^2+4} - \frac{s}{s^2+\omega^2}\right).
\end{aligned}$$

[1] Let's assume that $\omega > 0$.

(b): Assume that $\omega \neq 2$. Apply Laplace transforms to get

$$\begin{aligned} x'' + 4x &= \cos(\omega t) \\ s^2X + 4X &= \frac{s}{s^2+\omega^2} \qquad\qquad x(0) = x'(0) = 0 \\ (s^2+4)X &= \frac{s}{s^2+\omega^2} \\ X &= \frac{s}{(s^2+4)(s^2+\omega^2)}. \end{aligned}$$

Now apply part (a) to get

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}\left[\frac{s}{(s^2+4)(s^2+\omega^2)}\right] \\ &= L^{-1}\left[\frac{1}{\omega^2-4}\left(\frac{s}{s^2+4} - \frac{s}{s^2+\omega^2}\right)\right] \\ &= \frac{1}{\omega^2-4}\left(\mathscr{L}^{-1}\left[\frac{s}{s^2+4}\right] - \mathscr{L}^{-1}\left[\frac{s}{s^2+\omega^2}\right]\right) \\ &= \frac{1}{\omega^2-4}\left(\cos(2t) - \cos(\omega t)\right). \end{aligned}$$

(c): If $\omega = 2$ then the previous solution is wrong. We could find the solution by taking the limit as $\omega \to 2$:

$$x(t) = \lim_{\omega \to 2} \frac{\cos(2t) - \cos(\omega t)}{\omega^2 - 4}.$$

Instead we will use Laplace transforms:

$$\begin{aligned} x'' + 4x &= \cos(2t) \\ s^2X + 4X &= \frac{s}{s^2+4} \qquad\qquad x(0) = x'(0) = 0 \\ (s^2+4)X &= \frac{s}{s^2+4} \\ X &= \frac{s}{(s^2+4)^2}. \end{aligned}$$

Recall from Problem 1(a) that

$$\mathscr{L}[t \cdot \sin(2t)] = \frac{4s}{(s^2+4)^2}.$$

Hence

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}\left[\frac{s}{(s^2+4)^2}\right] \\ &= \frac{1}{4} \cdot \mathscr{L}^{-1}\left[\frac{4s}{(s^2+4)^2}\right] \\ &= \frac{1}{4} \cdot t \cdot \sin(2t). \end{aligned}$$

5. Hitting a Spring with a Hammer. The undamped oscillator $x''(t) + x(t) = 0$ with initial conditions $x(0) = 0$ and $x'(0) = 1$ has solution $x(t) = \sin t$. If we hit this spring with a hammer at time $t = a > 0$, then the equation becomes

$$x''(t) + x(t) = \delta(t-a); \quad x(0) = 0, x'(0) = 1.$$

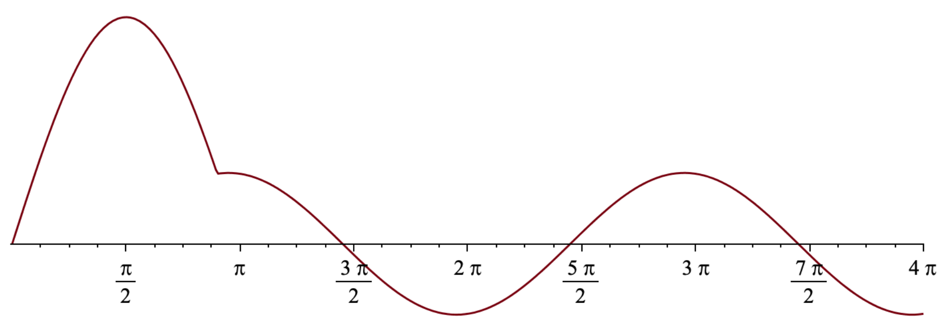
(a) Solve the new equation. [Hint: Use rule (5). Your answer will involve $H(t-a)$.]
(b) Use a computer to graph your solution for the following three values of a:

$$a = \frac{9\pi}{10}, \qquad a = \pi, \qquad a = \frac{11\pi}{10}.$$

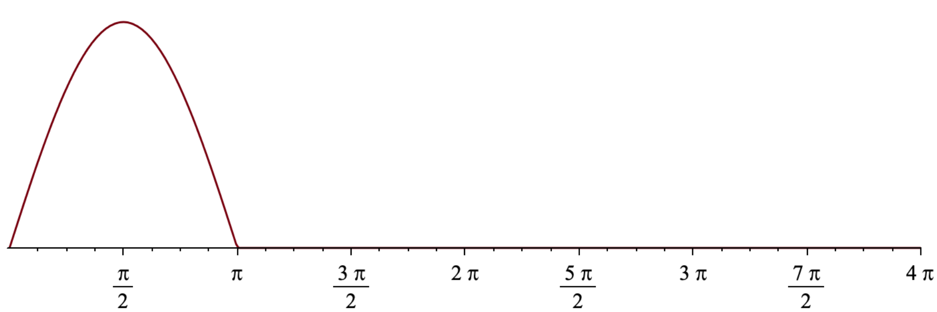
(a): Applying Laplace transforms gives

$$\begin{aligned}
x'' + x &= \delta(t-a) \\
s^2 X - sx(0) - x'(0) + X &= e^{-as} \\
s^2 - 1 + X &= e^{-as} \\
(s^2+1)X &= 1 + e^{-as} \\
X &= \frac{1}{s^2+1} + \frac{e^{-as}}{s^2+1} \\
x(t) &= \mathscr{L}^{-1}\left[\frac{1}{s^2+1}\right] + \mathscr{L}^{-1}\left[\frac{e^{-as}}{s^2+1}\right] \\
&= \sin(t) + H(t-a)\sin(t-a) \\
&= \begin{cases} \sin(t) & t < a, \\ \sin(t) + \sin(t-a) & t > a. \end{cases}
\end{aligned}$$

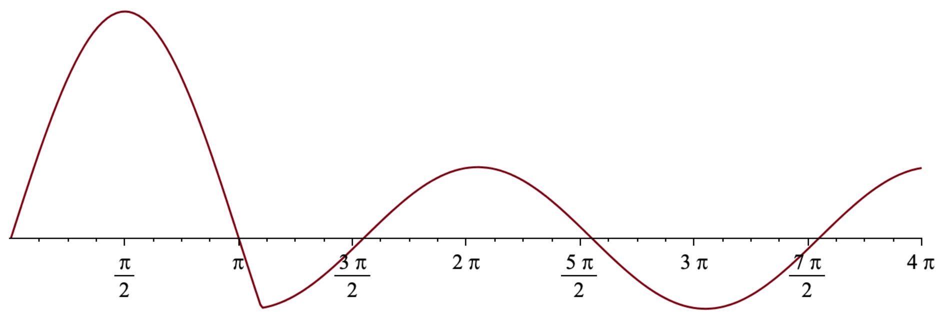
(b): Here is the graph of $x(t)$ when $a = 9\pi/10$:



Here is the graph of $x(t)$ when $a = \pi$:



Here is the graph of $x(t)$ when $a = 11\pi/10$:



Remark: We can simplify the formula for $x(t)$ by using the trig identity

$$\sin(\alpha+\beta)+\sin(\alpha-\beta) = 2\sin\alpha\cos\beta.$$

Putting $\alpha = t - a/2$ and $\beta = a/2$ gives

$$x(t) = \begin{cases} \sin(t) & t < a, \\ 2\cos(a/2)\sin(t-a/2) & t > a. \end{cases}$$

When $t < a$ the solution is a sine wave with amplitude 1. When $t > a$ the solution is a shifted sine wave with amplitude $2\cos(a/2)$.

6. Hockey Puck on Ice. A hockey puck of mass $m = 1$ sits on a flat sheet of ice with friction $\gamma > 0$. At time $t = 0$ a hockey stick instantaneously transfers 1 Newton of force to the puck. Let $x(t)$ be the horizontal distance of the puck from the hockey player, so that

$$x''(t) + \gamma \cdot x'(t) = \delta(t); \quad x(0) = x'(0) = 0.$$

(a) Find the partial fraction expansion of $\frac{1}{s(s+\gamma)}$.
(b) Solve for $x(t)$ in terms of γ. [Hint: Your answer will involve $H(t)$.]
(c) How far does the puck go before it is stopped by friction? [Hint: $\lim_{t\to\infty} x(t)$.]

(a): We are looking for A, B such that

$$\begin{aligned} \frac{1}{s(s+\gamma)} &= \frac{A}{s} + \frac{B}{s+\gamma} \\ \frac{1}{s(s+\gamma)} &= \frac{A(s+\gamma)+Bs}{s(s+\gamma)} \\ 1 &= A(s+\gamma)+Bs. \end{aligned}$$

Substituting $s = 0$ and $s = -\gamma$ gives $A = 1/\gamma$ and $B = -1/\gamma$, hence

$$\frac{1}{s(s+\gamma)} = \frac{1/\gamma}{s} + \frac{-1/\gamma}{s+\gamma} = \frac{1}{\gamma}\left(\frac{1}{s} - \frac{1}{s+\gamma}\right).$$

(b): Applying Laplace transforms and part (a) gives

$$\begin{aligned} x'' + \gamma x' &= \delta(t) \\ s^2X - sx(0) - x'(0) + \gamma(sX - x(0)) &= 1 \\ s^2X + s\gamma X &= 1 \qquad\qquad x(0) = x'(0) = 0 \\ s(s+\gamma)X &= 1 \end{aligned}$$

$$
\begin{aligned}
X &= \frac{1}{s(s+\gamma)} \\
x(t) &= \mathscr{L}^{-1}\left[\frac{1}{s(s+\gamma)}\right] \\
&= \mathscr{L}^{-1}\left[\frac{1}{\gamma}\left(\frac{1}{s} - \frac{1}{s+\gamma}\right)\right] \\
&= \frac{1}{\gamma} \cdot \mathscr{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+\gamma}\right] \\
&= \frac{1}{\gamma}\left(\mathscr{L}^{-1}\left[\frac{1}{s}\right] - \mathscr{L}^{-1}\left[\frac{1}{s+\gamma}\right]\right) \\
&= \frac{1}{\gamma}\left(1 - e^{-\gamma t}\right).
\end{aligned}
$$

(c): Since $\gamma > 0$ we have

$$\lim_{t\to\infty} \frac{1}{\gamma}(1 - e^{-\gamma t}) = \frac{1}{\gamma}(1-0) = \frac{1}{\gamma}.$$

That is, the puck will travel $1/\gamma$ units of distance before stopping. (Actually, it never completely stops, but the velocity decays rapidly to zero.)