Math 311 — Homework 3

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The **second order, linear, homogeneous ODE with constant coefficients** has the form

$$mx'' + \gamma x' + kx = 0.$$

We can think of this as a damped oscillator. The general solution will depend on two parameters, and a unique solution is determined by specifying the initial position $x(0)$ and velocity $x'(0)$.[1] To obtain the general solution, **we first look for basic solutions of the form $\mathbf{x(t) = e^{\lambda t}}$**. Substituting this guess into the ODE gives (after a bit if simplification) the *characteristic equation*

$$m\lambda^2 + \gamma\lambda + k = 0.$$

Let λ_1, λ_2 be the two roots of this equation. There are two cases:

- If $\lambda_1 \neq \lambda_2$ then the general solution is $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$.
- If $\lambda_1 = \lambda_2$ then the general solution is $x(t) = c_1 e^{\lambda 1 t} + c_2 t e^{\lambda_1 t}$.

If λ_1, λ_2 are **not real** then they must be complex conjugates: $\lambda_1, \lambda_2 = a \pm ib$ with $b \neq 0$, in which case Euler's formula allows us to express the solution in terms of sine and cosine:

$$c_1 e^{a+ib} + c_2 e^{a-ib} = e^{at}\left(c_3 \cos(bt) + c_4 \sin(bt)\right) \qquad \text{for some new constants } c_3, c_4.$$

After finding the general solution $x(t)$ we compute $x'(t)$ and then substitute $t = 0$ to obtain two equations for the two unknown constants, which determine the constants uniquely.

1. Distinct Real Roots. Solve the following equations:

(a) $y'' - 3y' + 2y = 0$ with $y(0) = 4$ and $y'(0) = 3$,
(b) $y'' - 4y = 0$ with $y(0) = 1$ and $y'(0) = 0$,
(c) $y'' - 3y' = 0$ with $y(0) = 5$ and $y'(0) = 3$.

(a): The independent variable wasn't named in the problem. For fun, let's call it t. We guess the basic solution $y(t) = e^{\lambda t}$ and substitute to get the characteristic equation:

$$\begin{aligned}
y''(t) - 3y'(t) + 2y(t) &= 0 \\
\lambda^2 e^{\lambda t} - 3\lambda e^{\lambda t} + 2e^{\lambda t} &= 0 \\
e^{\lambda t}(\lambda^2 - 3\lambda + 2) &= 0 \\
\lambda^2 - 3\lambda + 2 &= 0 \\
(\lambda - 1)(\lambda - 2) &= 0.
\end{aligned}$$

The two roots are $\lambda_1, \lambda_2 = 1, 2$ hence the general solution is

$$y(t) = c_1 e^{1t} + c_2 e^{2t}.$$

To determine c_1 and c_2 we first consider $y(t)$ and its derivative $y'(t)$:

$$\begin{cases} y(t) &= c_1 e^t + c_2 e^{2t}, \\ y'(t) &= c_1 e^t + 2c_2 e^{2t}. \end{cases}$$

Then we substitute $t = 0$ to get a system of two equations for c_1 and c_2:

$$\begin{cases} 4 &= c_1 + c_2, \\ 3 &= c_1 + 2c_2. \end{cases}$$

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[1] More generally, you can specify the position $x(t_1)$ and the velocity $x'(t_2)$ at any times t_1 and t_2.

Substituting these equations gives $c_2 = -1$ and back-substituting gives c_5. Hence

$$\boxed{y(t) = 5e^t - e^{2t}.}$$

(b): We guess the solution $y(t) = e^{\lambda t}$ and substitute to get the characteristic equation:

$$\begin{aligned} y''(t) - 4y(t) &= 0 \\ \lambda^2 e^{\lambda t} - 4e^{\lambda t} &= 0 \\ e^{\lambda t}(\lambda^2 - 4) &= 0 \\ \lambda^2 - 4 &= 0 \\ (\lambda + 2)(\lambda - 2) &= 0. \end{aligned}$$

The two roots are $\lambda_1, \lambda_2 = -2, 2$ hence the general solution is

$$y(t) = c_1 e^{-2t} + c_2 e^{2t}.$$

To determine c_1 and c_2 we first consider $y(t)$ and its derivative $y'(t)$:

$$\begin{cases} y(t) &= c_1 e^{-2t} + c_2 e^{2t}, \\ y'(t) &= -2c_1 e^{-2t} + 2c_2 e^{2t}. \end{cases}$$

Then we substitute $t = 0$ to get a system of two equations for c_1 and c_2:

$$\begin{cases} 1 &= c_1 + c_2, \\ 0 &= -2c_1 + 2c_2. \end{cases}$$

The second equation gives $c_1 = c_2$ then the first equation gives $c_1 = c_2 = 1/2$. Hence

$$\boxed{y(t) = e^{-2t}/2 + e^{2t}/2.}$$

Remark: This can also be expressed as $\cosh(2t)$.

(c): We guess the solution $y(t) = e^{\lambda t}$ and substitute to get the characteristic equation:

$$\begin{aligned} y''(t) - 3y'(t) &= 0 \\ \lambda^2 e^{\lambda t} - 3\lambda e^{\lambda t} &= 0 \\ e^{\lambda t}(\lambda^2 - 3\lambda) &= 0 \\ \lambda^2 - 3\lambda &= 0 \\ \lambda(\lambda - 3) &= 0. \end{aligned}$$

The two roots are $\lambda_1, \lambda_2 = 0, 3$ hence the general solution is

$$y(t) = c_1 e^{0t} + c_2 e^{3t} = c_1 + c_2 e^{3t}.$$

To determine c_1 and c_2 we first consider $y(t)$ and its derivative $y'(t)$:

$$\begin{cases} y(t) &= c_1 + c_2 e^{3t}, \\ y'(t) &= 0 + 3c_2 e^{3t}. \end{cases}$$

Then we substitute $t = 0$ to get a system of two equations for c_1 and c_2:

$$\begin{cases} 5 &= c_1 + c_2, \\ 3 &= 3c_2. \end{cases}$$

Solving this system gives $c_1 = 4$ and $c_2 = 1$, hence

$$\boxed{y(t) = 4 + e^{3t}.}$$

2. Repeated Roots. Solve the following equations:

(a) $y'' + 2y' + y = 0$ with $y(0) = 1$ and $y'(0) = 1$,
(b) $y'' = 0$ with $y(0) = 2$ and $y'(0) = 3$. [use the general method with repeated root $\lambda = 0$.]
(c) Now solve the equation $y'' = 0$ using two direct integrations. Observe that you get the same answer as with the general method.

(a): We guess the solution $y(t) = e^{\lambda t}$ and substitute to get the characteristic equation:

$$\begin{aligned} y''(t) + 2y(t) + y(t) &= 0 \\ \lambda^2 e^{\lambda t} + y(t) + y(t) &= 0 \\ e^{\lambda t}(\lambda^2 + 2\lambda + 1) &= 0 \\ \lambda^2 + 2\lambda + 1 &= 0 \\ (\lambda + 1)^2 &= 0. \end{aligned}$$

This time there is a repeated root $\lambda_1 = -1$. Hence the general solution is

$$y(t) = c_1 e^{-t} + c_2 t e^{-t}.$$

To determine c_1 and c_2 we must first compute $y'(t)$ via the chain rule:

$$\begin{aligned} y'(t) &= -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t} \\ &= -c_1 e^{-t} + c_2(1-t)e^{-t}. \end{aligned}$$

Thus we obtain the system

$$\begin{cases} y(t) &= c_1 e^{-t} + c_2 t e^{-t}, \\ y'(t) &= -c_1 e^{-t} + c_2(1-t)e^{-t}. \end{cases}$$

Substituting the initial conditions $y(0) = 1$ and $y'(0) = 1$ gives

$$\begin{cases} 1 &= c_1 + 0, \\ 1 &= -c_1 + c_2. \end{cases}$$

The first equation gives c_1, then the second equation gives $c_2 = 2$, hence

$$\boxed{y(t) = e^{-t} + 2te^{-t} = (1+2t)e^{-t}.}$$

(b): The equation $y''(t) = 0$ can be solved by direct integration. Integrating gives

$$\begin{aligned} y'(t) &= \int y''(t)\,dt = \int 0\,dt = c_1, \\ y(t) &= \int y'(t)\,dt = \int c_1\,dt + c_2 = c_1 t + c2, \end{aligned}$$

for some constants c_1 and c_2. Substituting $y(0) = 2$ and $y'(0) = 3$ gives $c_1 = 3$ and $c_2 = 2$, hence the solution is

$$\boxed{y(t) = 3t + 2.}$$

(c): But let's check that the general method still works. Substitute $y(t) = e^{\lambda t}$ to get the characteristic equation:

$$\begin{aligned} y''(t) &= 0 \\ \lambda^2 e^{\lambda t} &= 0 \\ \lambda^2 &= 0. \end{aligned}$$

We see that $\lambda = 0$ is a repeated root, hence the general solution is

$$y(t) = c_1 e^{0t} + c_2 t e^{0} = c_1 + c_2 t.$$

To determine c_1 and c_2 we consider the sytem:

$$\begin{cases} y(t) &= c_1 + c_2 t, \\ y'(t) &= c_2. \end{cases}$$

Substituting the initial conditions $y(0) = 2$ and $y'(0) = 3$ gives

$$\begin{cases} 2 &= c_1 + 0, \\ 3 &= c_2. \end{cases}$$

Hence the solution is

$$\boxed{y(t) = 2 + 3t.}$$

Yes, this agrees with part (b). Remark: Sometimes we have a choice between several methods of solution.

3. Complex Conjugate Roots. Solve the following equations. Express your answer in terms of sine and cosine:

(a) $y'' + y = 0$ with $y(0) = 2$ and $y'(0) = 3$,
(b) $y'' + 4y' + 13y = 0$ with $y(0) = 1$ and $y'(0) = 1$,
(c) $y'' + y' + y = 0$ with $y(0) = 0$ and $y'(0) = 3$.

(a): **This example is very important. We starting talking about it on the first day of class.** At that time we observed that $\sin t$ and $\cos t$ are solutions, and I told you that the general solution has the form

$$y(t) = A\cos t + B\sin t.$$

Let's check that this agrees with the general method. Substitute $y(t) = e^{\lambda t}$ to get the characteristic equation:

$$\begin{aligned} y''(t) + y(t) &= 0 \\ \lambda^2 e^{\lambda t} + e^{\lambda t} &= 0 \\ e^{\lambda t}(\lambda^2 + 1) &= 0 \\ \lambda^2 + 1 &= 0 \\ \lambda^2 &= -1 \\ \lambda &= \pm i. \end{aligned}$$

Since there are two distinct roots $\lambda_1, \lambda_2 = +i, -i$ the general solution is

$$y(t) = c_1 e^{it} + c_2 e^{-it}.$$

This doesn't look like $A\sin t + B\cos t$ but it turns out to be exactly the same. To see this we can use Euler's formulas:

$$\begin{aligned} e^{it} &= \cos t + i\sin t, \\ e^{-it} &= \cos(-t) + i\sin(-t) = \cos t - i\sin t. \end{aligned}$$

Substituting these into our solution gives

$$\begin{aligned} y(t) &= c_1 e^{it} + c_2 e^{-it} \\ &= c_1(\cos t + i\sin t) + c_2(\cos t - i\sin t) \\ &= (c_1 + c_2)\cos t + (ic_1 - ic_2)\sin t. \end{aligned}$$

Since $c_1 + c_2$ and $ic_1 - ic_2$ are just constants, we are free to rename them as $A = c_1 + c_2$ and $B = ic_1 - ic_2$.[2] In order to solve for A and B we first need the derivative:

$$y'(t) = (A\cos t + B\sin t)' = -A\sin t + B\cos t.$$

Then we substitute:

$$\begin{cases} y(t) & = A\cos t + B\sin t, \\ y'(t) & = -A\sin t + B\cos t, \end{cases} \Rightarrow \begin{cases} 2 & = A\cos 0 + B\sin 0, \\ 3 & = -A\sin 0 + B\cos 0, \end{cases} \Rightarrow \begin{cases} A = 2, \\ B = 3. \end{cases}$$

The final solution is

$$\boxed{y(t) = 2\cos t + 3\sin t.}$$

Remark: You should **memorize** the fact that $y''(t) = -y(t)$ implies $y(t) = A\cos t + B\sin t$.

(b): Substitute $y(t) = e^{\lambda t}$ to get the characteristic equation:

$$\begin{aligned} y''(t) + 4y'(t) + 13y(t) &= 0 \\ \lambda^2 e^{\lambda t} + 4\lambda e^{\lambda t} + 13e^{\lambda t} &= 0 \\ e^{\lambda t}(\lambda^2 + 4\lambda + 13) &= 0 \\ \lambda^2 + 4\lambda + 13 &= 0. \end{aligned}$$

We can solve this via the quadratic formula

$$\lambda = \frac{-4 \pm \sqrt{16 - 52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i.$$

The two roots are $\lambda_1, \lambda_2 = -2 \pm 3i$ hence the general solution is

$$\begin{aligned} y(t) &= c_1 e^{(-2+3i)t} + c_2 e^{(-2-3i)t} \\ &= c_1 e^{-2t+3it} + c_2 e^{-2t-3it} \\ &= c_1 e^{-2t} e^{i3t} + c_2 e^{-2t} e^{-3it} \\ &= e^{-2t}\left(c_1 e^{i3t} + c_2 e^{-i3t}\right). \end{aligned}$$

This formula is correct, but hard to interpret. We prefer to express the solution without using complex numbers. To do this we substitute Euler's formulas

$$\begin{aligned} e^{i3t} &= \cos(3t) + i\sin(3t), \\ e^{-i3t} &= \cos(-3t) + i\sin(-3t) = \cos(3t) - i\sin(3t), \end{aligned}$$

to obtain

$$\begin{aligned} y(t) &= e^{-2t}\left(c_1 e^{i3t} + c_2 e^{-i3t}\right) \\ &= e^{-2t}\left(c_1\left[\cos(3t) + i\sin(3t)\right] + c_2\left[\cos(3t) - i\sin(3t))\right]\right) \\ &= e^{-2t}\left(\left[c_1 + c_2\right]\cos(3t) + \left[c_1 i - c_2 i\right]\sin(3t)\right). \end{aligned}$$

Since $c_1 + c_2$ and $c_1 i - c_2 i$ are just constants, we are free to rename them. Let's say

$$y(t) = e^{-2t}\left(A\cos(3t) + B\sin(3t)\right).$$

This is the "real form" of the solution. The constants A, B are determined by the initial conditions, which were given as $y(0) = 1$ and $y'(0) = 1$. The condition $y(0) = 1$ tells us that

$$\begin{aligned} y(0) &= e^0\left(A\cos(0) + B\sin(0)\right) \\ 1 &= c_3. \end{aligned}$$

[2]There is a technical point here. To guarantee that this works, we need to know that the pair of functions e^{it} and e^{-it} is "linearly independent", as well as the pair $\cos t$ and $\sin t$. We'll discuss this concept later.

In order to substitute $y'(0) = 1$ we must first compute $y'(t)$, which requires the product rule:

$$y'(t) = -2e^{-2t}\left(A\cos(3t) + B\sin(3t)\right) + e^{-2t}\left(-3A\sin(3t) + 3B\cos(3t)\right).$$

(That was annoying.) Now we substitute $t = 0$ to get

$$\begin{aligned}
1 &= y'(0) \\
1 &= -2e^0\left(A\cos(0) + B\sin(0)\right) + e^0\left(-3A\sin(0) + 3B\cos(0)\right) \\
1 &= -2A + 3B \\
1 &= -2 \cdot 1 + 3B \\
3 &= 3B \\
1 &= B.
\end{aligned}$$

Hence the final solution is

$$\boxed{y(t) = e^{-2t}\left(\cos(3t) + \sin(3t)\right).}$$

(c): In parts (a) and (b) I showed all of the details. This time I will skip the routine parts. Substitute $y(t) = e^{\lambda t}$ to get the characteristic equation:

$$\begin{aligned}
y''(t) + y'(t) + y(t) &= 0 \\
&\vdots \\
\lambda^2 + \lambda + 1 &= 0.
\end{aligned}$$

We can solve this via the quadratic formula

$$\lambda_1, \lambda_2 = -\frac{1}{2} \pm \frac{3}{2}i.$$

Hence the general solution is

$$y(t) = c_1 e^{(-1/2 + i\sqrt{3}/2)t} + c_2 e^{(-1/2 - i\sqrt{3}/2})t.$$

This formula is correct, but we prefer to express it as

$$y(t) = e^{-t/2}\left(A\cos(t\sqrt{3}/2) + B\sin(t\sqrt{3}/2)\right).$$

To solve for A and B we first substitute $y(0) = 0$ to get

$$0 = y(0) = e^0(A\cos 0 + B\sin 0) = A.$$

This simplifies the formula to

$$y(t) = e^{-t/2}B\sin(t\sqrt{3}/2),$$

which makes it easier to solve for B. First we differentiate using the product rule:

$$y'(t) = -\frac{1}{2}e^{-t/2}B\sin(t\sqrt{3}/2) - \frac{\sqrt{3}}{2}B\cos(t\sqrt{3}/2).$$

Then substituting $y'(0) = 3$ gives

$$3 = y'(0) = 0 - \frac{\sqrt{3}}{2}B \quad \Longrightarrow \quad B = 2\sqrt{3}.$$

The final solution is

$$\boxed{y(t) = 2\sqrt{3} \cdot e^{-t/2} \cdot \sin(t\sqrt{3}/2).}$$

4. A Damped Oscillator. Consider the equation for a damped oscillator:

$$x''(t) + \gamma x'(t) + x(t) = 0,$$

where $\gamma \geq 0$ is the coefficient of friction. Solve the following problems for three different amounts of friction: $\gamma = 0, 1, 2$.

(a) Find the general form of the solution.
(b) Find the specific solution $x(t)$ with $x(0) = 0$ and $x'(0) = 1$.
(c) Graph the solution.

$\gamma = 0$: The equation $x''(t) + x(t) = 0$ has general solution $x(t) = A\cos t + B\sin t$, which you have memorized because I told you to. Substituting $x(0) = 0$ and $x'(0) = 1$ gives $A = 0$ and $B = 0$, hence

$$\boxed{x(t) = \sin t.}$$

$\gamma = 1$: In Problem 3(b) we saw that the equation $x''(t) + x'(t) + x(t) = 0$ has general solution

$$x(t) = e^{-t/2}\left(A\cos(t\sqrt{3}/2) + B\sin(t\sqrt{3}/2)\right).$$

Substituting initial conditions $x(0) = 0$ and $x'(0) = 1$ gives

$$\boxed{x(t) = \frac{2\sqrt{3}}{3} \cdot e^{-t/2} \cdot \sin\left(\frac{\sqrt{3}}{2}t\right).}$$

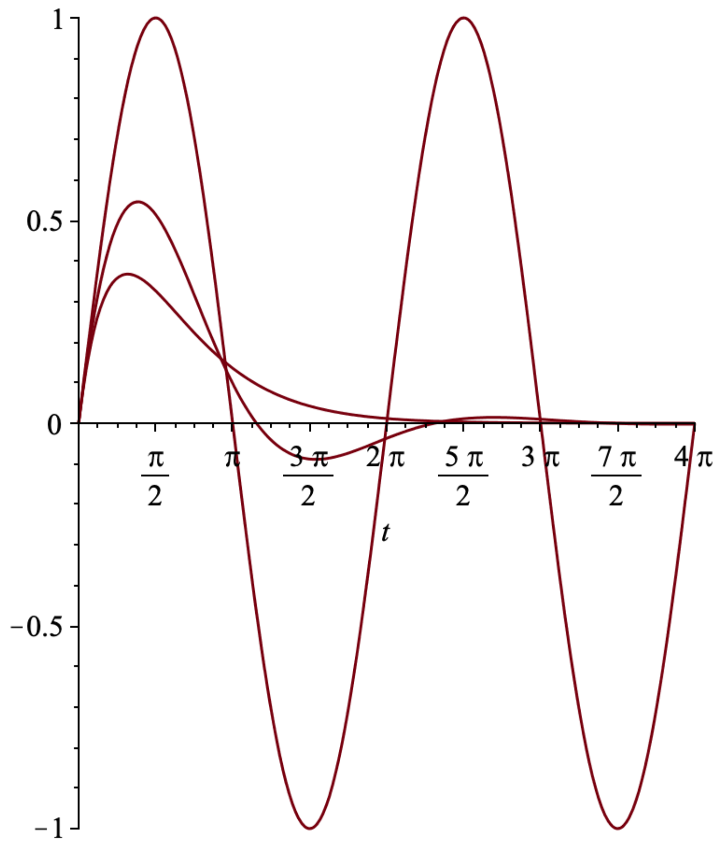
$\gamma = 2$: In Problem 2(a) we saw that the equation $x''(t) + 2x'(t) + x(t) = 0$ has general solution

$$x(t) = Ae^{-t} + Bte^{-t}.$$

Substituting $x(0) = 0$ and $x'(0) = 1$ gives $A = 0$ and $B = 1$, hence

$$\boxed{x(t) = te^{-t}.}$$

Here are the three graphs shown on the same axes:



Remarks:

- With no friction ($\gamma = 0$) the frequency is 1 and the amplitude is always 1.
- With a small amount of friction ($\gamma = 1$) there is still oscillation with a frequency of $\sqrt{3}/2$ but now the amplitude is $e^{-t/2} \cdot 2\sqrt{3}/3$, which quickly decays to zero.
- With a sufficient amount of friction ($\gamma = 2$) there is **no more oscillation**.
- The graph for any $0 < \gamma < 2$ looks roughly like the graph for $\gamma = 1$.
- The graph for any $\gamma \geq 2$ looks roughly like the graph for $\gamma = 2$.