Math 311 **Spring 2023**
Homework 3 **Drew Armstrong**

The **second order, linear, homogeneous ODE with constant coefficients** has the form

$$mx'' + \gamma x' + kx = 0.$$

We can think of this as a damped oscillator. The general solution will depend on two parameters, and a unique solution is determined by specifying the initial position $x(0)$ and velocity $x'(0)$.[1] To obtain the general solution, **we first look for basic solutions of the form** $x(t) = e^{\lambda t}$. Substituting this into the ODE gives (after a few steps) the characteristic equation

$$m\lambda^2 + \gamma\lambda + k = 0.$$

Let λ_1, λ_2 be the two roots of this equation. There are two cases:

- If $\lambda_1 \neq \lambda_2$ then the general solution is $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$.
- If $\lambda_1 = \lambda_2$ then the general solution is $x(t) = c_1 e^{\lambda 1 t} + c_2 t e^{\lambda_1 t}$.

If λ_1, λ_2 are **not real** then they must be complex conjugates: $\lambda_1, \lambda_2 = a \pm ib$ with $b \neq 0$, in which case Euler's formula allows us to express the solution in terms of sine and cosine:

$$c_1 e^{a+ib} + c_2 e^{a-ib} = e^{at}\left(c_3 \cos(bt) + c_4 \sin(bt)\right) \qquad \text{for some new constants } c_3, c_4.$$

After finding the general solution $x(t)$ we compute $x'(t)$ and then substitute $t = 0$ to obtain two equations for the two unknown constants, which determine the constants uniquely.

1. Distinct Real Roots. Solve the following equations:

(a) $y'' - 3y' + 2y = 0$ with $y(0) = 4$ and $y'(0) = 3$,
(b) $y'' - 4y = 0$ with $y(0) = 1$ and $y'(0) = 0$,
(c) $y'' - 3y' = 0$ with $y(0) = 5$ and $y'(0) = 3$.

2. Repeated Roots. Solve the following equations:

(a) $y'' + 2y' + y = 0$ with $y(0) = 1$ and $y'(0) = 1$,
(b) $y'' = 0$ with $y(0) = 2$ and $y'(0) = 3$. [use the general method with repeated root $\lambda = 0$.]
(c) Now solve the equation $y'' = 0$ using two direct integrations. Observe that you get the same answer as with the general method.

3. Complex Conjugate Roots. Solve the following equations. Express your answer in terms of sine and cosine:

(a) $y'' + y = 0$ with $y(0) = 2$ and $y'(0) = 3$,
(b) $y'' + 4y' + 13y = 0$ with $y(0) = 1$ and $y'(0) = 1$,
(c) $y'' + y' + y = 0$ with $y(0) = 0$ and $y'(0) = 3$.

4. A Damped Oscillator. Consider the equation for a damped oscillator:

$$x''(t) + \gamma x'(t) + x(t) = 0,$$

where $\gamma \geq 0$ is the coefficient of friction. Solve the following problems for three different amounts of friction: $\gamma = 0, 1, 2$.

(a) Find the general form of the solution.
(b) Find the specific solution $x(t)$ with $x(0) = 0$ and $x'(0) = 1$.
(c) Graph the solution.

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[1] More generally, you can specify the position $x(t_1)$ and the velocity $x'(t_2)$ at any times t_1 and t_2.