Math 311 **Spring 2023**
Homework 2 **Drew Armstrong**

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1. Integrating Factors for Linear ODEs. Solve the following equations for $y(x)$:

(a) $y' + y = e^x$ and $y(0) = 1$,
(b) $xy' + 2y = 3x$ and $y(1) = 5$,
(c) $xy' - y = x$ and $y(1) = 7$,
(d) $y' = 1 + 2xy$ and $y(0) = 5$. [Express your answer in terms of the error function

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2}\, ds.]$$

The general method: Consider an equation of the form

$$y'(x) + P(x)y(x) = Q(x).$$

Define the "integrating factor" $\rho(x) = \exp(\int P(x)\, dx)$, so that $\rho'(x) = \rho(x)P(x)$. Then we multiply both sides by $\rho(x)$ to obtain

$$\begin{aligned}
\rho(x)(y'(x) + P(x)y(x)) &= \rho(x)Q(x) \\
\rho(x)y'(x) + \rho(x)P(x)y(x) &= \rho(x)Q(x) \\
\rho(x)y'(x) + \rho'(x)y(x) &= \rho(x)Q(x) \\
[\rho(x)y(x)]' &= \rho(x)Q(x) \\
\rho(x)y(x) &= \int \rho(x)Q(x)\, dx + C \\
y(x) &= \frac{1}{\rho(x)} \left[\int \rho(x)Q(x)\, dx + C \right].
\end{aligned}$$

Instead of memorizing the final formula we will just apply the method in each separate case.

(a): The equation $y' + y = e^x$ has $P(x) = 1$ and $Q(x) = e^x$. The integrating factor is

$$\rho(x) = \exp\left(\int 1\, dx \right) = e^x.$$

Multiply both sides of the equation by e^x to get

$$\begin{aligned}
e^x(y' + y) &= e^x \cdot e^x \\
e^x y' + e^x y &= e^{2x} \\
(e^x y)' &= e^{2x}\, dx \\
e^x y &= \int e^{2x}\, dx + C \\
e^x y &= \frac{1}{2}e^{2x} + C \\
y &= \frac{1}{2}e^{2x}e^{-x} + Ce^{-x} \\
y &= \frac{1}{2}e^x + Ce^{-x}.
\end{aligned}$$

To determine C we substitute the initial condition $y(0) = 1$:

$$\begin{aligned} y(0) &= 1 \\ \frac{1}{2} + C &= 1 \\ C &= 1/2. \end{aligned}$$

So the solution is

$$\boxed{y = \frac{1}{2}e^x + \frac{1}{2}e^{-x} = \frac{e^x + e^{-x}}{2}.}$$

Remark: This function has a special name. It is called *hyperbolic cosine*:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}.$$

The graph of $\cosh(x)$ describes the shape of a hanging chain (or a free-standing arch, such as the St. Louis arch). It is related to the usual cosine function by replacing x with ix:

$$\cosh(ix) = \frac{e^{ix} + e^{-ix}}{2} = \cos(x).$$

(b): We put the equation $xy' + 2y = 3x$ in standard form

$$\begin{aligned} y' + \frac{2}{x}y &= 3 \\ y' + P(x)y &= Q(x), \end{aligned}$$

so that $P(x) = 2/x$ and $Q(x) = 3$. The integrating factor is

$$\rho(x) = \exp\left(\int 2/x\,dx\right) = \exp(2\ln(x)) = \exp(\ln(x^2)) = x^2.$$

Multiplying both sides by x^2 gives

$$\begin{aligned} x^2\left(y' + \frac{2}{x}y\right) &= 3x^2 \\ x^2y' + 2xy &= 3x^2 \\ (x^2y)' &= 3x^2 \\ x^2y &= \int 3x^2\,dx + C \\ x^2y &= x^3 + C \\ y &= x + C/x^2. \end{aligned}$$

To determine C we substitute the initial condition $y(1) = 5$:

$$\begin{aligned} y(1) &= 5 \\ 1 + C &= 5 \\ C &= 4. \end{aligned}$$

So the solution is

$$\boxed{y = x + \frac{4}{x^2}.}$$

(c): First we put $xy' - y = x$ in standard form:

$$\begin{aligned} y' - \frac{1}{x}y &= 1 \\ y' + P(x)y &= Q(x), \end{aligned}$$

so that $P(x) = -1/x$ and $Q(x) = 1$. The integrating factor is

$$\rho(x) = \exp\left(\int -1/x\,dx\right) = \exp(-\ln(x)) = \exp(\ln(1/x)) = 1/x.$$

Multiplying both sides by $1/x$ gives

$$\begin{aligned} \frac{1}{x}y' - \frac{1}{x^2}y &= \frac{1}{x} \\ \left(\frac{1}{x}\cdot y\right)' &= \frac{1}{x} \\ \frac{y}{x} &= \int \frac{1}{x}\,dx + C \\ \frac{y}{x} &= \ln(x) + C \\ y &= x(\ln(x) + C). \end{aligned}$$

To determine C we substitute $y(1) = 7$:

$$\begin{aligned} y(1) &= 7 \\ 1(\ln(1) + C) &= 7 \\ C &= 7. \end{aligned}$$

So the solution is

$$\boxed{y = x(\ln(x) + 7).}$$

(d): We put the equation $y' = 1 + 2xy$ in standard form:

$$\begin{aligned} y' - 2xy &= 1 \\ y' + P(x)y &= Q(x), \end{aligned}$$

with $P(x) = -2x$ and $Q(x) = 1$. The integration factor is

$$\rho(x) = \exp\left(\int -2x\,dx\right) = \exp(-x^2) = e^{-x^2}.$$

Multiplying both sides by e^{-x^2} gives

$$\begin{aligned} e^{-x^2}(y' - 2xy) &= e^{-x^2} \\ e^{-x^2}y' - 2xe^{-x^2}y &= e^{-x^2} \\ \left[e^{-x^2}y\right]' &= e^{-x^2} \\ e^{-x^2}y &= \int e^{-x^2}\,dx + C. \end{aligned}$$

Now we're stuck. At this point we should express the antiderivative as a definite integral. The lower bound is arbitrary because any change in lower bound can be absorbed into the constant C. So let's take lower bound 0:[1]

$$\begin{aligned} e^{-x^2} y &= \int_0^x e^{-s^2}\, ds + C \\ y &= e^{x^2} \int_0^x e^{-s^2}\, ds + Ce^{x^2} \\ y &= e^{x^2} \left[\int_0^x e^{-s^2}\, ds + C \right]. \end{aligned}$$

To determine C we substitute $y(0) = 5$:

$$\begin{aligned} y(0) &= 5 \\ e^0 \left[\int_0^0 e^{-s^2}\, ds + C \right] &= 5 \\ C &= 5. \end{aligned}$$

Hence the solution is

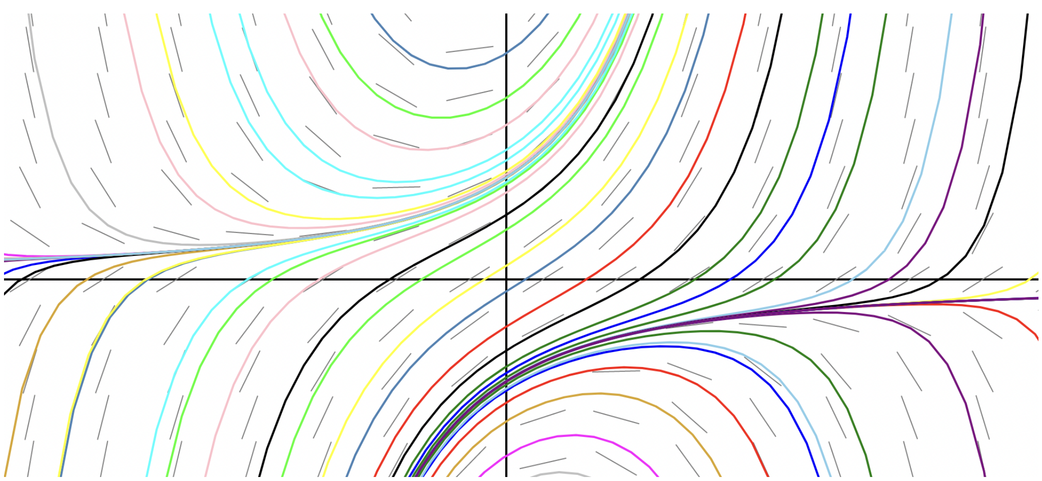
$$\boxed{y = e^{x^2} \left[\int_0^x e^{-s^2}\, ds + 5 \right].}$$

But we were asked to express this in terms of the error function, so we write

$$\boxed{y = e^{x^2} \left[\frac{\sqrt{\pi}}{2} \cdot \operatorname{erf}(x) + 5 \right].}$$

I know this is correct because my computer agrees.

Remark: I didn't ask you to think about the slope field, but here it is:



Note that some solutions go to $+\infty$ and some go to $-\infty$ as x goes to infinity. To be precise, my computer says that

$$y(x) = e^{x^2} \left[\frac{\sqrt{\pi}}{2} \cdot \operatorname{erf}(x) + y(0) \right] \to \begin{cases} +\infty & \text{if } y(0) > -1, \\ 0 & \text{if } y(0) = -1, \\ -\infty & \text{if } y(0) < -1. \end{cases}$$

[1] We choose 0 because the initial condition is given in terms of $y(0)$ and because the error function $\operatorname{erf}(x)$ is defined with lower bound 0. Convenient.

2. Logistic Growth with Harvesting. Let $x(t)$ be the size of a farmed population (maybe fish in a pond). Without harvesting, let's say the population has logistic growth $x'(t) = x(4-x)$. If we harvest the population at a constant rate $h > 0$ then we obtain the equation

$$x'(t) = x(4-x) - h, \quad \text{where } h > 0 \text{ is the constant rate of harvesting.}$$

Solve the following problems for three different rates of harvesting: $h = 3, 4, 5$.

(a) For which values of x is $x(4-x) - h$ positive, zero, negative?
(b) Use part (a) to sketch the slope field.
(c) Describe the behavior of $x(t)$ as $t \to \infty$. [Ignore negative solutions. If $x(t)$ becomes negative we say that the population is extinct.]

Remark: These equations can be solved exactly, but I'm not asking you to do that because the solutions are too complicated. Instead, we want a qualitative analysis.

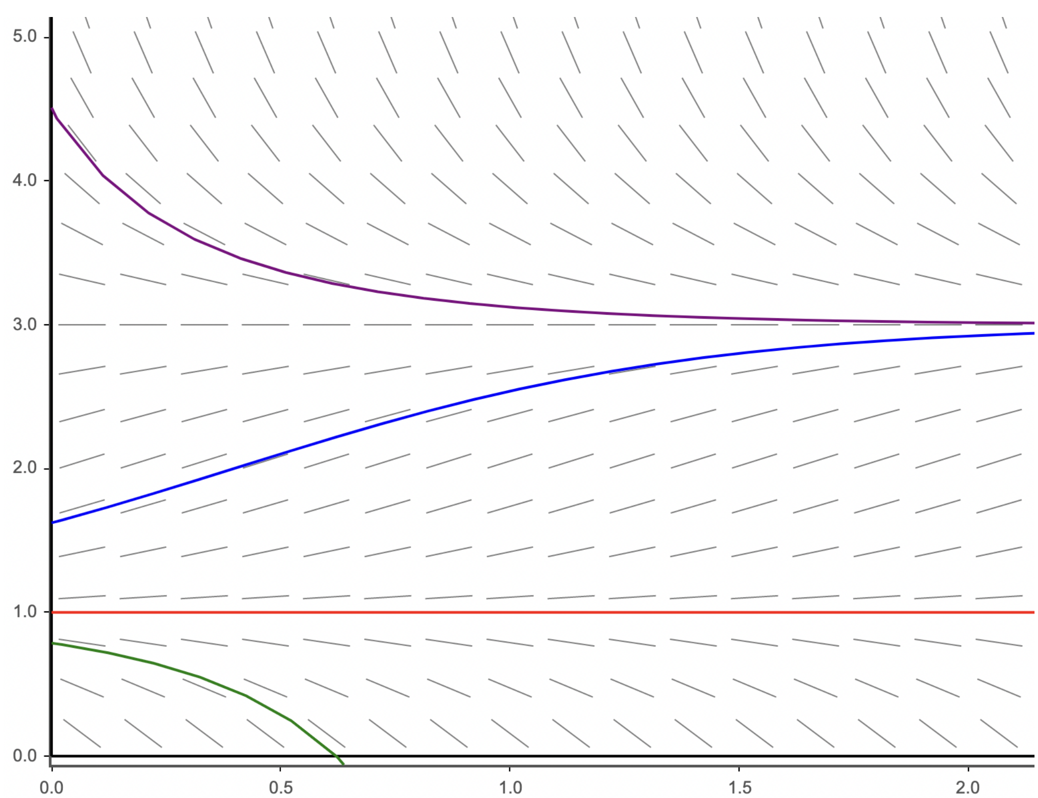
Solution for $h = 3$ (Underharvesting). The equation for the fish population is

$$x'(t) = x(4-x) - 3 = -x^2 + 4x - 3 = -(x-1)(x-3).$$

From this we see that

- $x'(t) = 0$ when $x = 1$ or $x = 3$,
- $x'(t) > 0$ when $1 < x < 3$,
- $x'(t) < 0$ when $x < 1$ or $3 < x$.

Here is the slope field with a few solution curves drawn:



If the initial population satisfies $x(0) > 1$ then the fish population stabilizes at 3. If the initial population satisfies $x(0) < 1$ then the fish go extinct. If $x(0) = 1$ then we have $x(t) = 1$ for all t, but this solution is unstable.

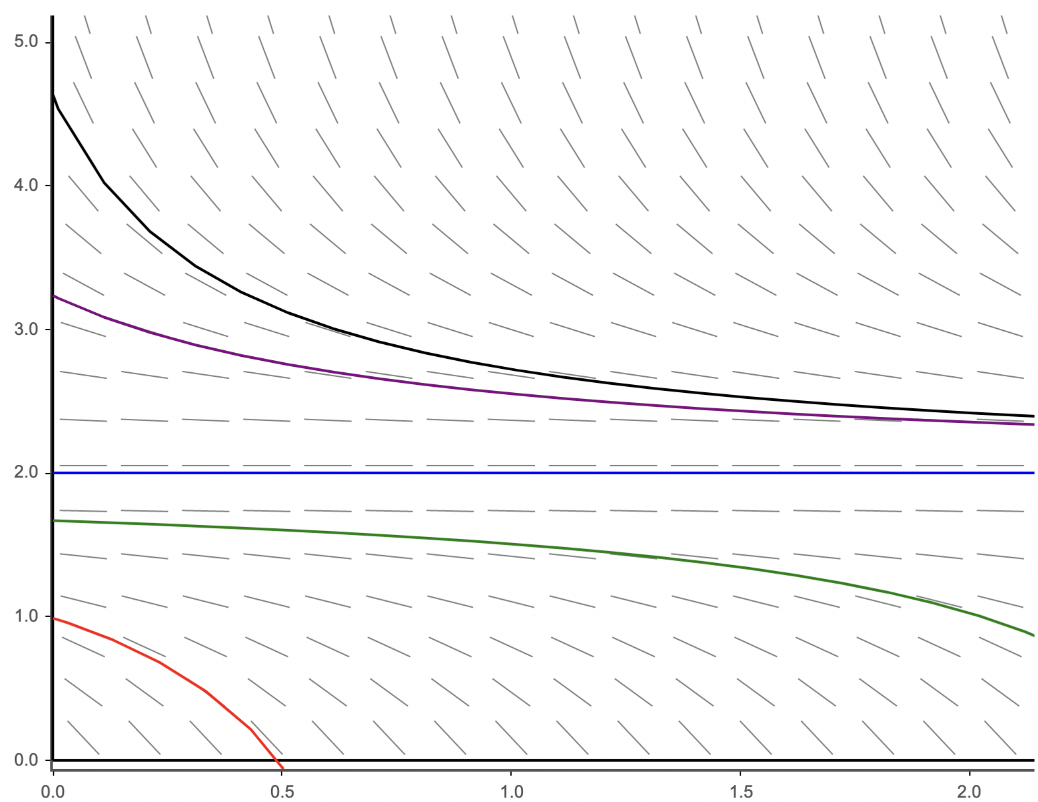
Solution for $h = 4$ (Critical Harvesting). The equation for the fish population is

$$x'(t) = x(4 - x) - 4 = -x^2 + 4x - 4 = -(x - 2)^2.$$

From this we see that

- $x'(t) = 0$ when $x = 2$,
- $x'(t) < 0$ when $x \neq 2$.

Here is the slope field with a few solution curves drawn:



If the initial population satisfies $x(0) \geq 2$ then the fish population stabilizes at 2. If the initial population satisfies $x(0) < 2$ then the fish go extinct.

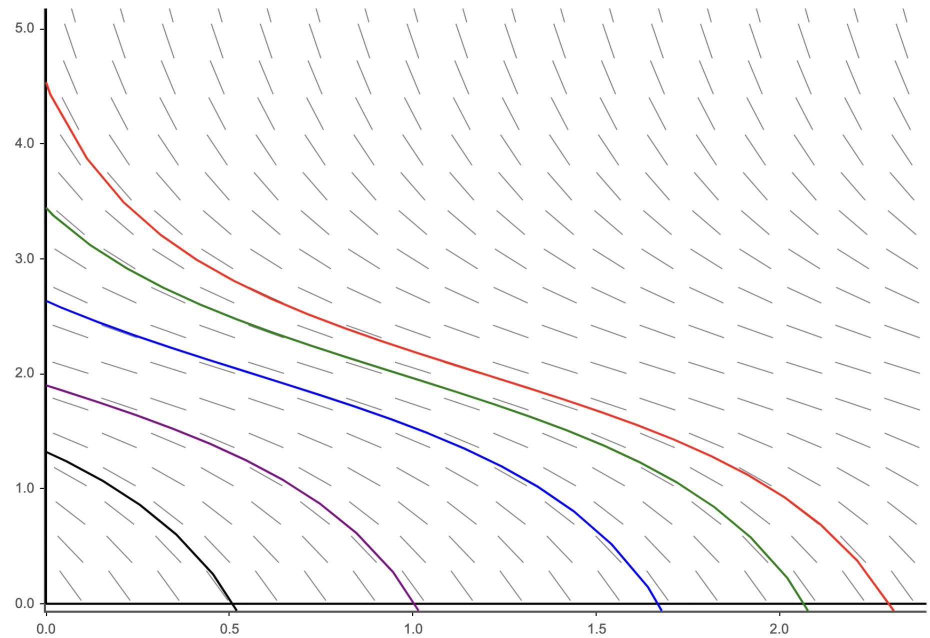
Solution for $h = 5$ (Overharvesting). The equation for the fish population is

$$x'(t) = x(4 - x) - 5 = -x^2 + 4x - 5.$$

This quadratic expression does not factor. And one can check that

- $x'(t) < 0$ for any value of x.

No matter what the initial population is, the fish will go extinct:



3. Phase Shift. The angle sum identity for cosine tells us that

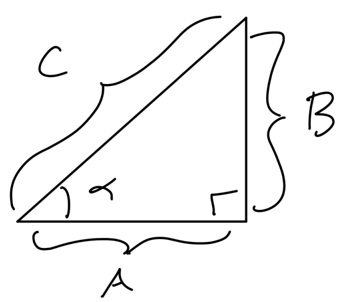
$$C\cos(x-\alpha) = C\cos\alpha\cos x + C\sin\alpha\sin x.$$

(a) Suppose that $C\cos(x-\alpha) = A\cos x + B\sin x$. Use the above identity to express C and α in terms of A and B. [Hint: We must have $A = C\cos\alpha$ and $B = C\sin\alpha$.]
(b) Use part (a) to express $\cos x + \sin x$ in the form $C\cos(x-\alpha)$.
(c) Graph the three functions $\cos x$, $\sin x$ and $C\cos(x-\alpha)$ on the same axes to make sure that your answer in part (b) makes sense.

(a): If $C\cos(x-\alpha) = A\cos x + B\sin x$ then from the trig identity we must have

$$A\cos x + B\sin x = (C\cos\alpha)\cos x + (C\sin\alpha)\sin x,$$

so that $A = C\cos\alpha$ and $B = C\sin\alpha$. At this point it is useful to draw a triangle:



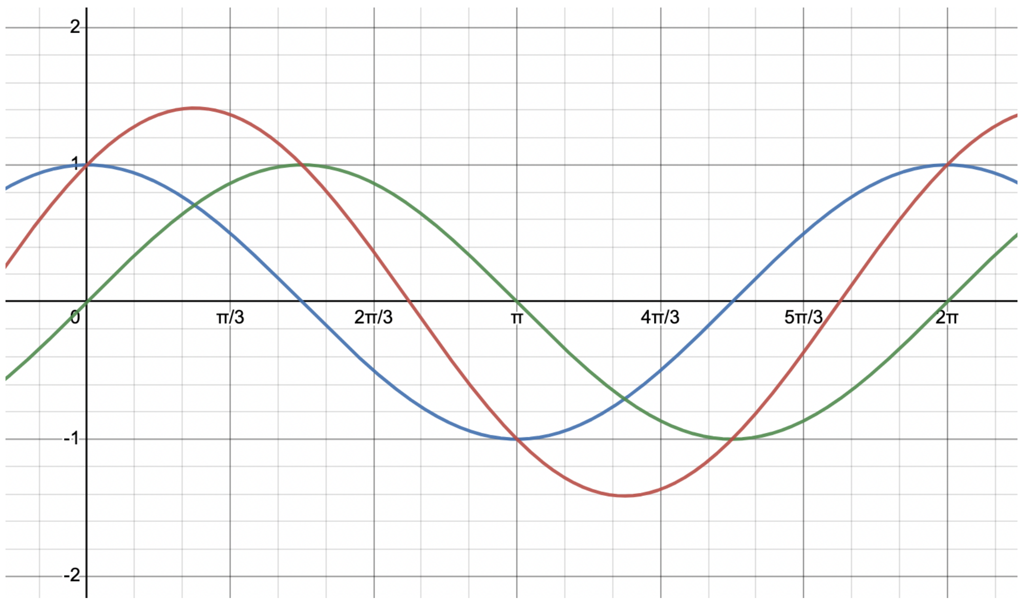
Then from the Pythagorean theorem and the definition of the tangent function we have

$$C = \sqrt{A^2+B^2},$$
$$\alpha = \tan^{-1}(B/A).$$

(b): When $A = B = 1$ we have $C = \sqrt{1^2+1^2} = \sqrt{2}$ and $\alpha = \tan^{-1}(1/1) = \pi/4$, so that

$$\cos x + \sin x = \sqrt{2}\cdot\cos\left(x - \frac{\pi}{4}\right).$$

(c): From the computer graph, we see that the heights of $\cos x$ and $\sin x$ add to the height of $\sqrt{2} \cdot \cos(x - \pi/4)$, as expected:[2]



4. Indoor vs Outdoor Temperature. We will use the function $\cos(t)$ to model the outdoor temperature. If $u(t)$ is the indoor temperature then Newton's Law says[3]

$$u'(t) = \cos(t) - u(t).$$

(a) Compute the general solution. [Hint: You will need the integral

$$\int e^t \cos t \, dt = \frac{e^t}{2} (\cos t + \sin t) + C.]$$

(b) Find the specific solution with $u(0) = 3$. Use a computer to graph the indoor temperature $u(t)$ and the outdoor temperature $\cos(t)$ on the same axes, say for $t = 0 \ldots 15$.

(c) As $t \to \infty$ the indoor temperature settles down to a simple oscillation. Compute the phase shift between the indoor and outdoor temperatures. After the outdoor temperature peaks, how many hours until the indoor temperature peaks? [Assume the outdoor temperature has a period of 24 hours.]

(a): This is a linear first order equation. We put the equation in standard form:

$$u'(t) + u(t) = \cos t$$
$$u'(t) + P(t)u(t) = Q(t),$$

where $P(t) = 1$ and $Q(t) = \cos t$. The integrating factor is

$$\rho(t) = \exp\left(\int P(t) \, dt\right) = \exp(t) = e^t.$$

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[2]Desmos labeled the x-axis with multiples of $\pi/3$, but multiples of $\pi/4$ would be more appropriate here.

[3]Technically, there should be some insulation constant $k > 0$ so that $u'(t) = k(\sin(t) - u(t))$. I took $k = 1$ for simplicity. We assume no air conditioning.

Multiply both sides by the integrating factor and solve:

$$\begin{aligned} u'(t) + u(t) &= \cos t \\ e^t u'(t) + e^t u(t) &= e^t \cos t \\ (e^t u(t))' &= e^t \cos t \\ e^t u(t) &= \int e^t \cos t \, dt + C \\ e^t u(t) &= \frac{e^t}{2}(\cos t + \sin t) + C \qquad \text{given} \\ u(t) &= \frac{1}{2}(\cos t + \sin t) + Ce^{-t}. \end{aligned}$$

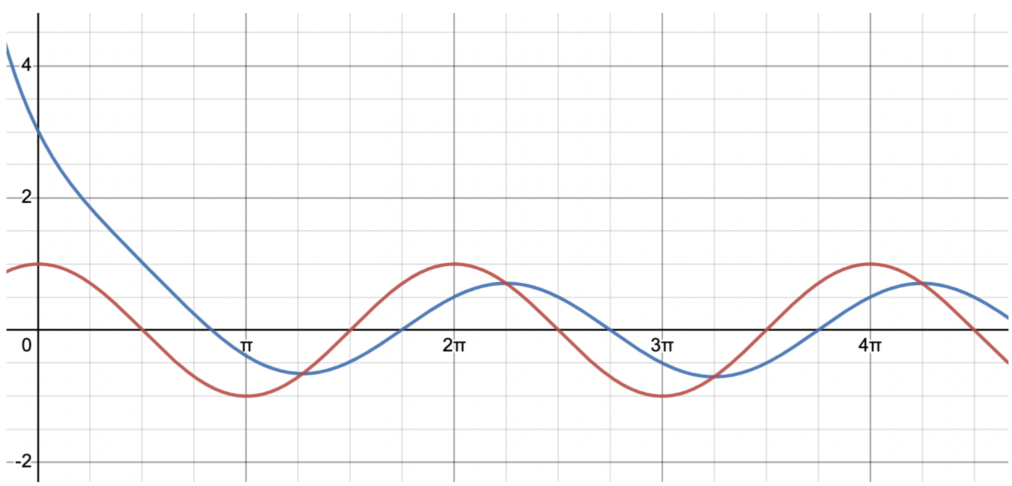
(b): The determine C we substitute the initial condition $u(0) = 3$:

$$\begin{aligned} u(0) &= 3 \\ \frac{1}{2}(\cos 0 + \sin 0) + Ce^0 &= 3 \\ \frac{1}{2} + C &= 3 \\ C &= 2.5. \end{aligned}$$

Hence the indoor temperature at time t is

$$\boxed{u(t) = \frac{1}{2}(\cos t + \sin t) + 2.5e^{-t}.}$$

Here is a graph of the outdoor temperature $\cos t$ versus the indoor temperature $u(t)$:



The transient term $2.5e^{-t}$ rapidly goes to zero and we are left with a steady state solution:

$$u(t) \approx \frac{1}{2}(\cos t + \sin t).$$

(c): In order to interpret the steady state, we must compute the amplitude and phase shift:

$$\frac{1}{2}\cos t + \frac{1}{2}\sin t = C\cos(t - \alpha).$$

Using the formulas from Problem 3(a) with $A = B = 1/2$ gives

$$C = \sqrt{A^2 + B^2} = \sqrt{1/4 + 1/4} = \sqrt{1/2} \approx 0.7,$$
$$\alpha = \tan^{-1}(B/A) = \tan^{-1}(1) = \pi/4.$$

Thus the steady state of the indoor temperature is

$$u(t) \approx 0.7 \cdot \cos(t - \pi/4).$$

This lags the outdoor temperature by $\pi/4$, which is $1/8$ of the full period 2π. If we view the period as 24 hours then **the lag is 3 hours**. That is, the indoor temperature will peak 3 hours after the outdoor temperature peaks. Does this make sense? We also note that the amplitude of the indoor temperature is ≈ 0.7. This is smaller than the amplitude of the outdoor temperature, which is 1. Does this make sense?

Remark: More generally, there is an insulation constant $k > 0$ so that

$$u'(t) = k\left(\cos t - u(t)\right).$$

In this case one can show that the time lag is $\tan^{-1}(1/k^2)$. Large values of k (bad insulation) cause a short time lag. Values of k close to zero (good insulation) cause a long time lag. Does this make sense?

5. Hooke's Law. I claim that the differential equation $x''(t) = -\omega^2 x(t)$ has general solution

$$x(t) = A\cos(\omega t) + B\sin(\omega t),$$

where A and B are arbitrary constants.

(a) Verify that this is, indeed, a solution.
(b) Solve for A and B in terms of the initial conditions $x(0)$ and $x'(0)$.
(c) The solution can alternatively be expressed as

$$x(t) = C\cos(\omega(t - \alpha)).$$

Solve for C and α in terms of $x(0)$ and $x'(0)$. [Hint: We can use the same method as in Problem 3. It is based on the angle sum identity:

$$\cos(\omega(t-\alpha)) = \cos(\omega t - \omega\alpha) = \cos(\omega\alpha)\cos(\omega t) + \sin(\omega\alpha)\sin(\omega t).]$$

(a): We saw in class that $\cos(\omega t)$ and $\sin(\omega t)$ are solutions. More generally, we will show that the formula $x(t) = A\cos(\omega t) + B\sin(\omega t)$ satisfies the differential equation $x''(t) = -\omega^2 x(t)$. First we compute $x'(t)$:

$$\begin{aligned} x'(t) &= \frac{d}{dx}\left[A\cos(\omega t) + B\sin(\omega t)\right] \\ &= A\frac{d}{dx}\cos(\omega t) + B\frac{d}{dx}\sin(\omega t) \\ &= A(-\omega\sin(\omega t)) + B\omega\cos(\omega t) \\ &= -A\omega\sin(\omega t) + B\omega\cos(\omega t). \end{aligned}$$

Then we compute $x''(t)$:

$$\begin{aligned}
x''(t) &= \frac{d}{dx}\left[-A\omega\sin(\omega t) + B\omega\cos(\omega t)\right] \\
&= -A\omega\frac{d}{dx}\sin(\omega t) + B\omega\frac{d}{dx}\cos(\omega t) \\
&= -A\omega\cdot\omega\cos(\omega t) + B\omega(-\omega\sin(\omega t)) \\
&= -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t) \\
&= -\omega^2\left[A\cos(\omega t) + B\sin(\omega t)\right],
\end{aligned}$$

and we observe that $x''(t) = -\omega^2 x(t)$ as desired.

Remark: The differential equation $x''(t) = -\omega^2 x(t)$ is **linear**, so the sum of any two solutions is also a solution. This is why we can add the solutions $A\cos(\omega t)$ and $B\sin(\omega t)$ to get another solution. We will say more about this later.

(b): To determine A and B, we substitute $t = 0$ into $x(t)$ and $x'(t)$ to get

$$\begin{cases} x(0) &= A\cos(0) + B\sin(0) &= A, \\ x'(0) &= -A\omega\sin(0) + B\omega\cos(0) &= B\omega. \end{cases}$$

Hence the solution in terms of the initial position and velocity is

$$\boxed{x(t) = x(0)\cos(\omega t) + \frac{x'(0)}{\omega}\sin(\omega t).}$$

(c): We want to find the amplitude C and phase shift α:

$$x(t) = x(0)\cos(\omega t) + \frac{x'(0)}{\omega}\sin(\omega t) = C\cos(\omega(t-\alpha)).$$

We will use the cosine difference of angles formula:

$$\begin{aligned}
A\cos(\omega t) + B\sin(\omega t) &= C\cos(\omega(t-\alpha)) \\
&= C\cos(\omega t - \omega\alpha) \\
&= C\left[\cos(\omega\alpha)\cos(\omega t) + \sin(\omega\alpha)\sin(\omega t)\right] \\
&= \left[C\cos(\omega\alpha)\right]\cos(\omega t) + \left[C\sin(\omega\alpha)\right]\sin(\omega t).
\end{aligned}$$

This implies that $A = C\cos(\omega\alpha)$ and $B = C\sin(\omega\alpha)$, hence

$$\begin{aligned}
C &= \sqrt{A^2 + B^2}, \\
\omega\alpha &= \tan^{-1}(B/A).
\end{aligned}$$

This is the same as in Problem 3, but using angle $\omega\alpha$ instead of α. In our case we have $A = x(0)$ and $B = x'(0)/\omega$, so that

$$\boxed{C = \sqrt{x(0)^2 + \left[\frac{x'(0)}{\omega}\right]^2} \quad \text{and} \quad \alpha = \frac{1}{\omega}\tan^{-1}\left(\frac{x'(0)/\omega}{x(0)}\right).}$$

Remark: Thus we have solved the general (undamped, unforced) harmonic oscillator with frequency $\omega = \sqrt{k/m}$, where m is the mass and k is the stiffness. I know it was a lot of algebra. This is a computation you should do exactly once in your life.

6. Euler's Identity. Let i denote a[4] square root of -1. *Euler's identity* provides a connection between exponential and trigonometric functions:

$$\boxed{e^{it} = \cos t + i \sin t.}$$

(a) Use Euler's identity to prove the *angle sum formulas*:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

[Hint: Use the property $e^{i\alpha}e^{i\beta} = e^{i\alpha+i\beta} = e^{i(\alpha+\beta)}$ of exponentials.]

(b) Use Euler's identity to prove that

$$\cos t = \frac{e^{it}+e^{-it}}{2} \quad \text{and} \quad \sin t = \frac{e^{it}-e^{-it}}{2i}.$$

[Hint: First show that $e^{-it} = \cos t - i \sin t$.]

(c) We have seen that the equation $x''(t) = -x(t)$ has general solution

$$x(t) = x(0)\cos t + x'(0)\sin t.$$

I claim that we can also express this solution in the form

$$x(t) = Ae^{it} + Be^{-it}$$

for some constants A and B. Use the formulas in part (b) to solve for A and B in terms of $x(0)$ and $x'(0)$. Your answers will involve imaginary numbers.

(a): On the one hand, we have

$$e^{i\alpha}\cdot e^{i\beta} = e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta).$$

On the other hand, we have

$$\begin{aligned} e^{i\alpha}\cdot e^{i\beta} &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= \cos\alpha\cos\beta + i\cos\alpha\sin\beta + i\sin\alpha\cos\beta + i^2\sin\alpha\sin\beta \\ &= \cos\alpha\cos\beta + i\cos\alpha\sin\beta + i\sin\alpha\cos\beta - \sin\alpha\sin\beta \\ &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta). \end{aligned}$$

Comparing the real and imaginary parts of the two expressions for $e^{i\alpha}\cdot e^{i\beta}$ gives the desired formulas.

(b): First we note that

$$e^{-it} = e^{i(-t)} = \cos(-t) + i\sin(-t) = \cos t - i\sin t.$$

Then we have

$$e^{it} + e^{-it} = (\cos t + \cancel{i\sin t}) + (\cos t - \cancel{i\sin t}) = 2\cos t$$

and

$$e^{it} - e^{-it} = (\cancel{\cos t} + i\sin t) - (\cancel{\cos t} - i\sin t) = 2i\sin t,$$

as desired.

[4] There are two square roots of -1. Pick your favorite and call it i. Then the other is called $-i$.

(c): We have seen that the equation $x''(t) = -x(t)$ has general solution $x(t) = x(0)\cos t + x'(0)\sin t$. To express this in the form $Ae^{it} + Be^{-it}$ we substitute the formulas from part (b):

$$\begin{aligned} x(t) &= x(0)\cos t + x'(0)\sin t \\ &= x(0)\left(\frac{e^{it} + e^{-it}}{2}\right) + x'(0)\left(\frac{e^{it} - e^{-it}}{2i}\right) \\ &= \left(\frac{x(0)}{2} + \frac{x'(0)}{2i}\right)e^{it} + \left(\frac{x(0)}{2} - \frac{x'(0)}{2i}\right)e^{-it}. \end{aligned}$$

We can simplify this a bit by using the fact that $1/i = -i$:

$$\boxed{x(t) = \left(\frac{x(0) - ix'(0)}{2}\right)e^{it} + \left(\frac{x(0) + ix'(0)}{2}\right)e^{-it}.}$$

Remark: This expression has lots of imaginary numbers in it, but these imaginary numbers somehow cancel to give the real solution $x(t) = x(0)\cos t + x'(0)\sin t$. So why bother with imaginary numbers? **Because they make computations easier!** (Well, maybe not today. But eventually they do.)