Math 311 **Spring 2023**
Homework 1 **Drew Armstrong**

The general first order ODE has the form

$$\frac{dy}{dx} = f(x,y) \quad \big(\text{or sometimes } x'(t) = f(x,t)\big)\,.$$

We can think of $f(x,y)$ as the slope of a tiny line at the point (x,y) in the x,y-plane. For any given point (a,b) (sometimes called the *initial condition*), if the function $f(x,y)$ is "nice" near $(x,y) = (a,b)$ then near this point there exists a unique solution $y(x)$ satisfying $y(a) = b$. We might not be able to write a formula for $y(x)$.

1. Direct Integration. Solve the equation $dy/dx = f(x,y)$ in the following cases:

(a) $f(x,y) = 2x+1$ and $y(0) = 3$,
(b) $f(x,y) = (x-2)^2$ and $y(2) = 1$,
(c) $f(x,y) = e^{-x^2}$ and $y(3) = 5$. [Hint Check that the following[1] is a solution: $y(x) = \int_3^x e^{-s^2}\,ds + C$ for some constant C. What is the correct value of C?]

(a): The general form of the solution is

$$\begin{aligned} dy/dx &= 2x+1 \\ y &= \int (2x+1)\,dx + C \\ y &= x^2 + x + C. \end{aligned}$$

To find the value of C we plug in the initial condition $y(0) = 3$:

$$\begin{aligned} y(0) &= 3 \\ 0^2 + 0 + C &= 3 \\ C &= 3. \end{aligned}$$

Hence the solution is $\boxed{y(x) = x^2 + x + 3}$.

(b): The general form of the solution is

$$\begin{aligned} dy/dx &= (x-2)^2 \\ y &= \int (x-2)^2\,dx + C \\ y &= (x-2)^3/3 + C. \end{aligned}$$

To find the value of C we plug in the initial condition $y(2) = 1$:

$$\begin{aligned} y(2) &= 1 \\ (2-2)^3/3 + C &= 1 \\ C &= 1. \end{aligned}$$

Hence the solution is $\boxed{y(x) = (x-2)^3/3 + 1}$.

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[1] Remark: This integral cannot be simplified.

(c): The general form of the solution is

$$\begin{aligned} dy/dx &= e^{-x^2} \\ y &= \int e^{-x^2}\, dx + C. \end{aligned}$$

This integral doesn't simplify. To find the constant C we need to be more specific. Actually, the solution is a definite integral

$$y = \int_{x_0}^{x} e^{-s^2}\, ds + C,$$

where the lower limit x_0 is arbitrary. (Changing the value of x_0 just changes the value of C.) Now we plug in the initial condition $y(3) = 5$:

$$\begin{aligned} y(3) &= 5 \\ \int_{x_0}^{3} e^{-s^2}\, dx + C &= 5 \\ C &= 5 - \int_{x_0}^{3} e^{-s^2}\, ds. \end{aligned}$$

As I said, the value of x_0 is arbitrary. However, we see that choosing $x_0 = 3$ vastly simplifies the solution:

$$C = 5 - \int_{3}^{3} e^{-s^2}\, dx = 5 - 0 = 5.$$

Hence the solution is

$$\boxed{y(x) = \int_{3}^{x} e^{-s^2}\, ds + 5.}$$

Remark: If you give this problem to your computer, it might respond using the *erf function*:[2]

$$\text{erf}(x) := \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-s^2}\, ds.$$

Note that $\frac{\sqrt{\pi}}{2}\text{erf}(x)$ is an antiderivative of e^{-x^2}. Thus we can represent the integral of e^{-x^2} between any bounds x_0 and x_1 by

$$\int_{x_0}^{x_1} e^{-s^2}\, ds = \frac{\sqrt{\pi}}{2}\left[\text{erf}(x_1) - \text{erf}(x_0)\right].$$

In this language, our solution becomes

$$\boxed{y(x) = \frac{\sqrt{\pi}}{2}\left[\text{erf}(x) - \text{erf}(3)\right] + 5.}$$

2. Separation of Variables. Solve the equation $dy/dx = f(x, y)$ in the following cases:

(a) $f(x, y) = y^2$ and $y(0) = 1$,
(b) $f(x, y) = y \cdot \sin x$ and $y(0) = 2$.
(c) $f(x, y) = xy + x + y + 1$ and $y(0) = 0$. [Hint: This equation doesn't look separable, but it is. Factor the expression for $f(x, y)$.]

[2] Short for "error function". This is the most important function in statistics, where is it also called $\Phi(x)$.

(a): First we apply separation of variables:

$$\begin{aligned}
\frac{dy}{dx} &= y^2 \\
\frac{dy}{y^2} &= dx \\
\int \frac{1}{y^2}\,dy &= x + C \\
-\frac{1}{y} &= x + C.
\end{aligned}$$

Then we solve for y:

$$y = -\frac{1}{x+C}.$$

Finally, we solve for C by plugging in the initial condition $y(0) = 1$:

$$\begin{aligned}
y(0) &= 1 \\
-\frac{1}{0+C} &= 1 \\
-1/C &= 1 \\
C &= -1.
\end{aligned}$$

Hence the solution is

$$\boxed{y(x) = -\frac{1}{x-1} = \frac{1}{1-x}.}$$

(b): First we use separation of variables:

$$\begin{aligned}
\frac{dy}{dx} &= y \cdot \sin(x) \\
\frac{dy}{y} &= \sin(x)\,dx \\
\int \frac{1}{y}\,dy &= \int \sin(x)\,dx + C \\
\ln(y) &= -\cos(x) + C \\
y &= e^{-\cos(x)+C} \\
y &= e^{-\cos(x)} \cdot e^{C}.
\end{aligned}$$

We usually prefer to rename the constant $D = e^C$:

$$y = De^{-\cos(x)}.$$

To solve for D we plug in the initial condition $y(0) = 2$:

$$\begin{aligned}
y(0) &= 2 \\
De^{-\cos(0)} &= 2 \\
De^{-1} &= 2 \\
D &= 2e.
\end{aligned}$$

Hence the solution is $y(x) = 2e \cdot e^{-\cos(x)}$, which looks better as

$$\boxed{y(x) = 2e^{-\cos(x)+1}.}$$

So I guess we shouldn't have renamed the constant. Then would have found $C = 1$. (Silly textbook problem.)

(c): First we factor and then use separation of variables:

$$\begin{aligned}
\frac{dy}{dx} &= xy + x + y + 1 \\
\frac{dy}{dx} &= (x+1)(y+1) \\
\frac{dy}{y+1} &= (x+1)dx \\
\int \frac{1}{y+1}\, dx &= \int (x+1)\, dx + C \\
\ln(y+1) &= x^2/2 + x + C \\
y + 1 &= \exp\left(x^2/2 + x + C\right) \\
y &= \exp(x^2/2 + x) \cdot \exp(C) - 1 \\
y &= D\exp(x^2/2 + x) - 1.
\end{aligned}$$

Again I renamed the constant $D = \exp(C)$. To solve for D we input $y(0) = 0$:

$$\begin{aligned}
y(0) &= 0 \\
D\exp(0) - 1 &= 0 \\
D &= 1.
\end{aligned}$$

Hence the solution is

$$\boxed{y(x) = \exp(x^2/2 + x) - 1.}$$

3. Logistic Growth. Let $x(t)$ be the size of a population of bacteria at time t. In the presence of limited resources, this population might follow the logistic growth equation:

$$x'(t) = x(1000 - x).$$

(a) Find the general form of the solution. [Hint: Use partial fractions to write $1/[x(1000-x)]$ in the form $A/x + B/(1000-x)$ for some constants A and B.]
(b) Sketch the solution with initial population size $x(0) = 1$.

(a): First we find the partial fractions:

$$\frac{1}{x(1000-x)} = \frac{A}{x} + \frac{B}{(1000-x)} = \frac{A(1000-x) + Bx}{x(1000-x)}.$$

Comparing numerators gives $1 = A(1000-x) + Bx$. Putting $x = 1000$ gives $B = 1/1000$ and putting $x = 0$ gives $A = 1/1000$. Now we integrate:

$$\begin{aligned}
\int \frac{1}{x(1000-x)} &= \int \left(\frac{A}{x} + \frac{B}{(1000-x)}\right) dx \\
&= A\ln(x) - B\ln(1000-x) && \text{(careful the negative sign)} \\
&= \ln(x)/1000 - \ln(1000-x)/1000
\end{aligned}$$

$$= \frac{1}{1000} \ln\left(\frac{x}{1000-x}\right).$$

Finally, we use separation of variables:

$$\begin{aligned}
dx/dt &= x(1000-x) \\
\frac{dx}{x(1000-x)} &= dt \\
\frac{1}{1000}\ln\left(\frac{x}{1000-x}\right) &= t + C \\
\ln\left(\frac{x}{1000-x}\right) &= 1000t + D && (D = 1000C) \\
\frac{x}{1000-x} &= e^{1000t+D} \\
\frac{x}{1000-x} &= Ee^{1000t} && (E = e^D) \\
x &= Ee^{-1000t}(1000-x) \\
x &= 1000Ee^{1000t} - xEe^{1000t} \\
x(1 + Ee^{1000t}) &= 1000Ee^{1000t} \\
x &= 1000Ee^{1000t}/(1 + Ee^{1000t}).
\end{aligned}$$

Actually, this looks nicer if we multiply top and bottom by e^{-1000t}/E:

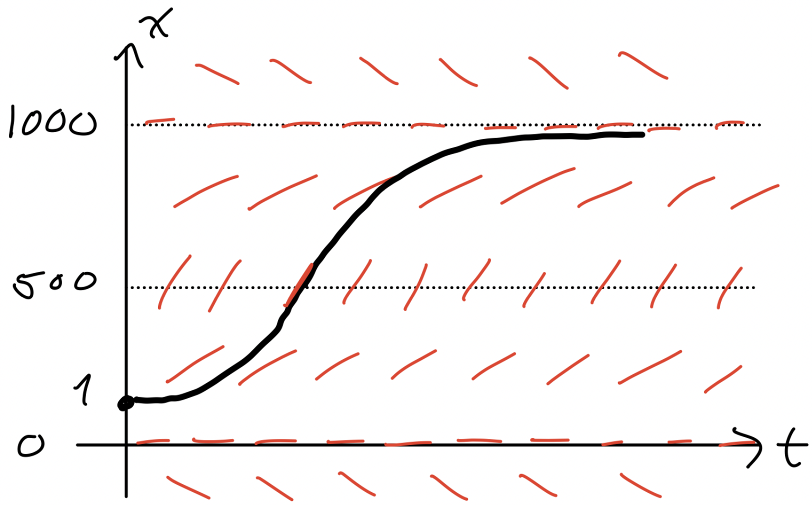
$$\boxed{x(t) = \frac{1000}{1 + Fe^{-1000t}}.}$$

Here I renamed $F = 1/E$.

(b): There are two ways to sketch the solution. First we can sketch the slope field. Note that the slope $x'(t) = x(1000-x)$ is:

- zero when $x = 0$ or $x = 1000$,
- negative when $x < 0$,
- positive when $0 < x < 1000$,
- negative when $x < 0$ or $1000 < x$,
- maximum when $x = 500$.

So the slope field looks roughly like this:

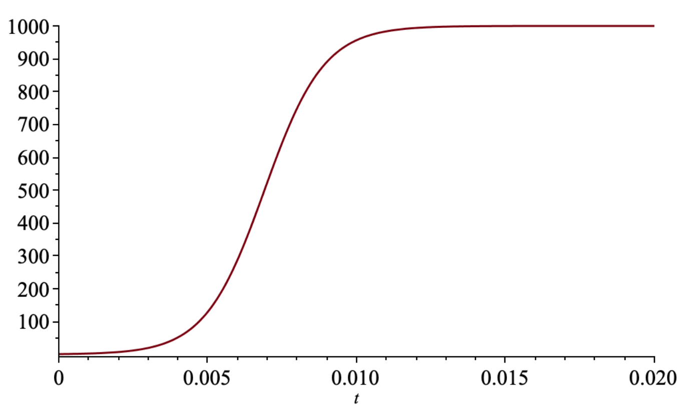


Note: There is an inflection point when $x = 500$ because the slope is maximized at this point.

Or we could explicitly solve the equation and graph it based on the formula (either by hand or with a computer). To find the constant F we plug in the initial condition $x(0) = 1$:

$$\begin{aligned} x(x) &= 1 \\ \frac{1000}{1 + Fe^0} &= 1 \\ \frac{1000}{1 + F} &= 1 \\ F &= 999. \end{aligned}$$

Hence the solution is $\boxed{x(t) = 1000/(1 + 999e^{-1000t})}$. My computer drew the graph:



Note that $x(t)$ grows **very quickly** (exponentially) before leveling off at 1000. This is the typical shape of logistic growth. It is sometimes called an "S-curve".

4. Free Fall. A projectile of mass m is launched straight up, near the surface of a planet. Let $h(t), v(t)$ and $a(t)$ denote the height, velocity and acceleration at time t. Let $h_0 = h(0)$ and $v_0 = v(0)$ denote the initial height and initial velocity (pointed upwards).

(a) In the absence of air resistance, Galileo says that $a(t) = -g$ for some positive constant $g > 0$. Use direct integration twice to find a formula for $h(t)$ in terms of the initial conditions h_0 and v_0, and the constant g.

(b) In the presence of air resistance we must modify Galileo's law to $a(t) = -g - \rho v(t)$ for some positive constant $\rho > 0$. We can also write this as $dv/dt = -g - \rho v$. Use separation of variables to solve for $v(t)$ in terms of v_0, g and ρ.

(c) *Terminal Velocity.* Use your solution from (b) to show that $v(t)$ approaches a constant as $t \to \infty$. Find a formula for this constant in terms of g and ρ.

(a): Recall that $v(t) = h'(t)$ and $a(t) = v'(t) = h''(t)$. Galileo says that $h''(t) = a(t) = -g$ is constant. We integrate twice to get

$$\begin{aligned}
h''(t) &= -g \\
h'(t) &= -gt + C_1 \\
h(t) &= -\frac{1}{2}gt^2 + C_1 t + C_2,
\end{aligned}$$

for some constants C_1 and C_2. (Since this is a second order equation there are two parameters in the the general solution.) Now we use the initial conditions to solve for C_1 and C_2. Putting $v(0) = v_0$ gives

$$\begin{aligned}
v(0) &= v_0 \\
h'(0) &= v_0 \\
-g(0) + C_1 &= v_0 \\
C_1 &= v_0.
\end{aligned}$$

And putting $h(0) = h_0$ gives

$$\begin{aligned}
h(0) &= h_0 \\
-\frac{1}{g}(0)^2 + v_0(0) + C_2 &= h_0 \\
C_2 &= h_0.
\end{aligned}$$

Hence the solution is

$$\boxed{h(t) = -\frac{1}{2}gt^2 + v_0 t + h_0.}$$

We can use this equation to solve any question about the height. For example, the quadratic formula allows us to find when the projectile hits the ground:

$$\begin{aligned}
h(t) &= 0 \\
-\frac{1}{2}gt^2 + v_0 t + h_0 &= 0 \\
t &= \frac{-v_0 \pm \sqrt{v_0^2 + 2gh_0}}{-1}.
\end{aligned}$$

(b): In the presence of air resistance, a more realistic model is $a(t) = -g - \rho v(t)$, which is the same as

$$\begin{aligned}
v'(t) &= -g - \rho v(t) \\
\frac{dv}{dt} &= -g - \rho v.
\end{aligned}$$

We can solve this by separation of variables:

$$\begin{aligned}
\frac{dv}{-g-\rho v} &= dt \\
\int \frac{1}{-g-\rho v}\,dv &= t + C \\
-\frac{1}{\rho}\ln(-g-\rho v) &= t + C && \text{(I computed the integral with } u = -g-\rho v) \\
\ln(-g-\rho v) &= -\rho t + D && (D = -\rho C) \\
-g-\rho v &= e^{-\rho t + D} \\
-g-\rho v &= Ee^{-\rho t} && (E = e^D) \\
v &= \frac{Ee^{-\rho t}+g}{-\rho}.
\end{aligned}$$

To find the value of E we plug in the initial velocity $v(0) = v_0$:

$$\begin{aligned}
v(0) &= v_0 \\
\frac{Ee^{-\rho(0)}+g}{-\rho} &= v_0 \\
\frac{E+g}{-\rho} &= v_0 \\
E &= -\rho v_0 - g.
\end{aligned}$$

Hence the solution is

$$\boxed{v(t) = \frac{(-\rho v_0 - g)e^{-\rho t}+g}{-\rho}}.$$

(c): Since $e^{-\rho t} \to 0$ as $t \to \infty$ (because $\rho > 0$), we find that

$$v(t) = \frac{(-\rho v_0 - g)e^{-\rho t}+g}{-\rho} \to \frac{0+g}{-\rho} = -\frac{g}{\rho} \quad \text{as} \quad t \to \infty.$$

The constant g represents the mass of the planet. The constant ρ represents the air resistance from the atmosphere of the planet. If you jump out of a plane then your velocity will increase until the gravity and air resistance perfectly balance. At this point your velocity will be constant: $v(t) \approx -g/\rho$. That is, until you hit the ground or open a parachute. (Opening a parachute increases ρ, so decreases the terminal velocity $|g/\rho|$.)

5. Newton's Law of Cooling. Let $u(t)$ be the temperature of a cup of coffee at time t and let A be the ambient temperature of the room. Newton's Law of Cooling[3] says that

$$\frac{du}{dt} = A - u.$$

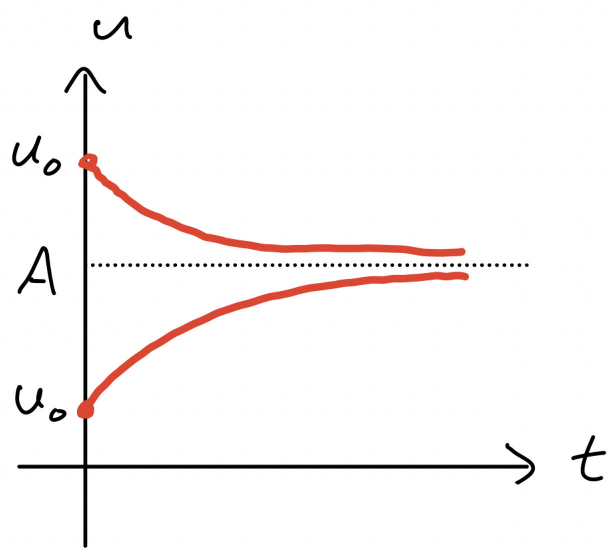
(a) Solve for $u(t)$ in terms of A and the initial temperature $u_0 = u(0)$.
(b) Sketch the graphs of two solutions in the t, u-plane: one solution with $0 < u_0 < A$ and one solution with $A < u_0$.
(c) Now suppose that the ambient temperature is **not constant**; let's say $A = t$, so the temperature is increasing with time. Then Newton's Law becomes $du/dt = t - u$. This equation cannot be solved with our current methods. Instead, sketch the slope field in the t, u-plane and sketch a typical solution with $u_0 > 0$.

[3] Technically, we have $du/dt = k(A - u)$ for some positive constant $k > 0$. We will set $k = 1$ for simplicity.

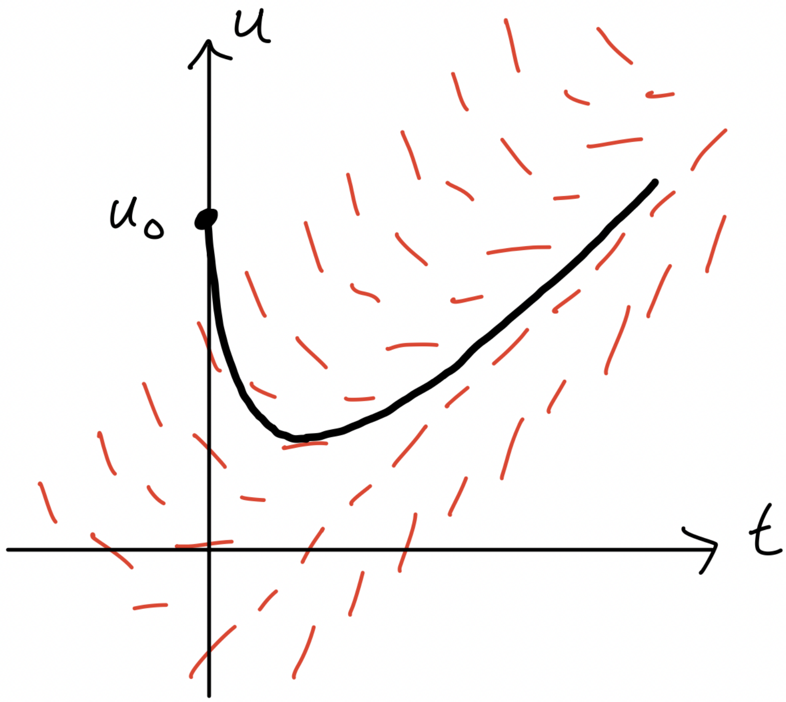
(a): We use separation of variables to find the general solution:

$$\begin{aligned}
du/dt &= A - u \\
\frac{du}{A-u} &= dt \\
\int \frac{1}{A-u}\,du &= t + C \\
-\ln(A-u) &= t + C \\
\ln(A-u) &= -t + D && (D = -C) \\
A - u &= e^{-t+D} \\
A - u &= Ee^{-t} && (E = e^{D}) \\
u &= A - Ee^{-t}.
\end{aligned}$$

(b): The graph of $u(t)$ looks essentially like e^{-t} but it approaches A as $t \to \infty$. If the coffee begins hotter than the room ($u_0 > A$) then it gradually cools down. If the coffee begins cooler than the room ($u_0 < A$) then it gradually heats up:



(c): Now the room is heating up at a constant rate: $A(t) = t$. The temperature of the coffee still satisfies Newton's equation $u'(t) = A(t) - u(t) = t - u$. When I assigned this problem we did not have the tools to solve it exactly. But we can still sketch the slope field and graph a typical solution with $u_0 > 0$:



Note that the coffee first cools down (because the room starts at $A = 0$), but then it heats up as the room heats up.

Remark: But now we do have the tools to solve it exactly. We are dealing with a **linear** equation:

$$\frac{du}{dt} + u = t$$
$$\frac{du}{dt} + P(t)u = Q(t),$$

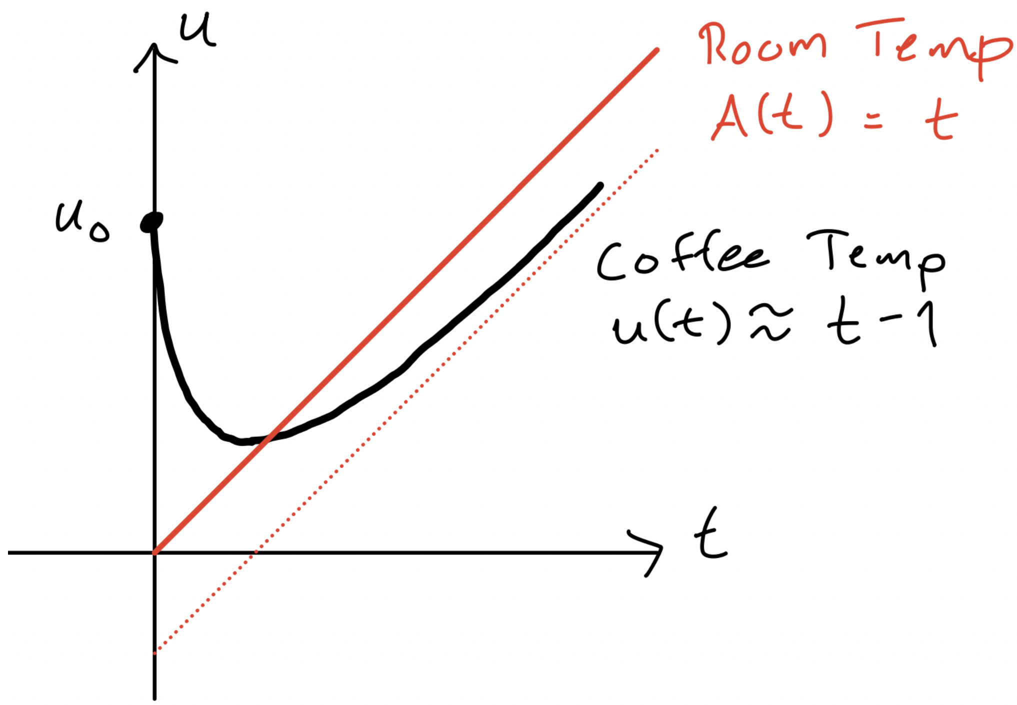
where $P(t) = 1$ and $Q(t) = t$. The integrating factor for this equation is

$$\rho(t) = e^{\int P(t)\,dt} = e^t.$$

Multiply both sides by $\rho(t)$ and then integrate. The left hand side simplifies via the product rule, because this is what integrating factors are **designed** to do:

$$\begin{aligned}
e^t(u' + u) &= e^t t \\
e^t u' + e^t u &= te^t \\
(e^t \cdot u)' &= te^t \\
e^t u &= \int te^t\,dt + C \\
e^t u &= (t-1)e^t + C && \text{(details omitted)} \\
u &= t - 1 + Ce^{-t}.
\end{aligned}$$

Note that $u(t) \approx t - 1$ for large t. Thus $u(t)$ approaches the straight line $u = t - 1$. Physically: The coffee starts hotter than the room. It cools down until it becomes slightly colder than the room. Then the room and the coffee heat up at the same rate, but the temperature of the coffee lags the temperature of the room by one unit of time. Picture:



6. A Non-Separable Equation. Consider the first order ODE

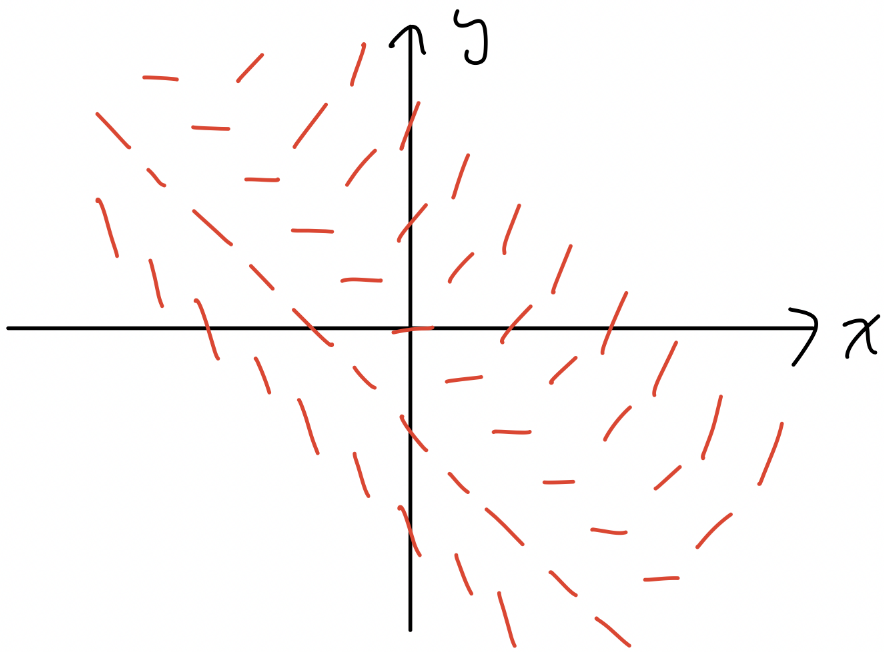
$$\frac{dy}{dx} = f(x, y) = x + y.$$

(a) Sketch the slope field in the x, y-plane.
(b) One of the solution curves is a straight line. Find the equation of this line. [Hint: Suppose that $y = mx + b$ for some constants m and b. Then we have

$$m = \frac{dy}{dx} = x + y = x + mx + b.$$

Hence the equation $m = x + mx + b$ holds for any value of x. Use this to solve for m and b.]

(a): This problem is very much like Problem 5(c):



(b): From the picture we guess that one of the solution curves is a straight line. That is, one of the solution curves has the form $y(x) = mx + b$ for some constants m, b. We can find this special solution without having to find the general solution of $dy/dx = x + y$. In order to find m and b we plug $y(x) = mx + b$ into the differential equation:

$$\begin{aligned} dy/dx &= x + y \\ \frac{d}{dx}(mx + b) &= x + (mx + b) \\ m &= x + mx + b. \end{aligned}$$

This equation must hold for all x. Putting $x = 0$ gives $m = b$ and putting $x = 1$ gives $b = -1$. Hence the straight line solution is

$$y(x) = -x - 1.$$

Note that this agrees with our sketch of the slope field.

Remark: When I assigned the homework we did not know how to find the general solution of $dy/dx = x + y$. Now we can solve it with the method of integrating factors. The general solution is

$$y(x) = -x - 1 + Ce^x.$$

Note that this is just the special solution plus the general solution to the associated "homogeneous" equation $dy/dx = y$. We will see that this phenomenon is common to all "linear" differential equations.