Math 311 **Spring 2023**
Homework 1 **Drew Armstrong**

The general first order ODE has the form

$$\frac{dy}{dx} = f(x,y) \quad \big(\text{or sometimes } x'(t) = f(x,t)\big).$$

We can think of $f(x,y)$ as the slope of a tiny line at the point (x,y) in the x,y-plane. For any given point (a,b) (sometimes called the *initial condition*), if the function $f(x,y)$ is "nice" near $(x,y) = (a,b)$ then near this point there exists a unique solution $y(x)$ satisfying $y(a) = b$. We might not be able to write a formula for $y(x)$.

1. Direct Integration. Solve the equation $dy/dx = f(x,y)$ in the following cases:

(a) $f(x,y) = 2x + 1$ and $y(0) = 3$,
(b) $f(x,y) = (x-2)^2$ and $y(2) = 1$,
(c) $f(x,y) = e^{-x^2}$ and $y(3) = 5$. [Hint Check that the following[1] is a solution: $y(x) = \int_3^x e^{-s^2}\,ds + C$ for some constant C. What is the correct value of C?]

2. Separation of Variables. Solve the equation $dy/dx = f(x,y)$ in the following cases:

(a) $f(x,y) = y^2$ and $y(0) = 1$,
(b) $f(x,y) = y \cdot \sin x$ and $y(0) = 2$.
(c) $f(x,y) = xy + x + y + 1$ and $y(0) = 0$. [Hint: This equation doesn't look separable, but it is. Factor the expression for $f(x,y)$.]

3. Logistic Growth. Let $x(t)$ be the size of a population of bacteria at time t. In the presence of limited resources, this population might follow the logistic growth equation:

$$x'(t) = x(1000 - x).$$

(a) Find the general form of the solution. [Hint: Use partial fractions to write $1/[x(5-x)]$ in the form $A/x + B/(1000-x)$ for some constants A and B.]
(b) Sketch the solution with initial population size $x(0) = 1$.

4. Free Fall. A projectile of mass m is launched straight up, near the surface of a planet. Let $h(t), v(t)$ and $a(t)$ denote the height, velocity and acceleration at time t. Let $h_0 = h(0)$ and $v_0 = v(0)$ denote the initial height and initial velocity (pointed upwards).

(a) In the absence of air resistance, Galileo says that $a(t) = -g$ for some positive constant $g > 0$. Use direct integration twice to find a formula for $h(t)$ in terms of the initial conditions h_0 and v_0, and the constant g.
(b) In the presence of air resistance we must modify Galileo's law to $a(t) = -g - \rho v(t)$ for some positive constant $\rho > 0$. We can also write this as $dv/dt = -g - \rho v$. Use separation of variables to solve for $v(t)$ in terms of v_0, g and ρ.
(c) *Terminal Velocity.* Use your solution from (b) to show that $v(t)$ approaches a constant as $t \to \infty$. Find a formula for this constant in terms of g and ρ.

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[1] Remark: This integral cannot be simplified.

5. Newton's Law of Cooling. Let $u(t)$ be the temperature of a cup of coffee at time t and let A be the ambient temperature of the room. Newton's Law of Cooling[2] says that

$$\frac{du}{dt} = A - u.$$

(a) Solve for $u(t)$ in terms of A and the initial temperature $u_0 = u(0)$.

(b) Sketch the graphs of two solutions in the t, u-plane: one solution with $0 < u_0 < A$ and one solution with $A < u_0$.

(c) Now suppose that the ambient temperature is **not constant**; let's say $A = t$, so the temperature is increasing with time. Then Newton's Law becomes $du/dt = t - u$. This equation cannot be solved with our current methods. Instead, sketch the slope field in the t, u-plane and sketch a typical solution with $u_0 > 0$.

6. A Non-Separable Equation. Consider the first order ODE

$$\frac{dy}{dx} = f(x, y) = x + y.$$

(a) Sketch the slope field in the x, y-plane.

(b) One of the solution curves is a straight line. Find the equation of this line. [Hint: Suppose that $y = mx + b$ for some constants m and b. Then we have

$$m = \frac{dy}{dx} = x + y = x + mx + b.$$

Hence the equation $m = x + mx + b$ holds for any value of x. Use this to solve for m and b.]

[2] Technically, we have $du/dt = k(A - u)$ for some positive constant $k > 0$. We will set $k = 1$ for simplicity.