Math 311 | **Final Exam**
Spring 2023 | **Fri May 5**

No electronic devices are allowed. No collaboration is allowed. There are 10 pages and each page is worth 6 points, for a total of 60 points.

1. Direct Integration. Consider the equation $x''(t) = t$.

(a) Find the general solution.

Integrating twice gives

$$\begin{aligned} x''(t) &= t \\ x'(t) &= \frac{1}{2}t^2 + c_1 \\ x(t) &= \frac{1}{6}t^3 + c_1 t + c_2 \end{aligned}$$

for some constants c_1, c_2.

(b) Find the solution with $x(0) = 2$ and $x'(0) = 3$.

Substituting $t = 0$ in $x'(t)$ gives

$$\begin{aligned} x'(0) &= \frac{1}{2}0^2 + c_1 \\ 3 &= c_1, \end{aligned}$$

and substituting $t = 0$ in $x(0)$ gives

$$\begin{aligned} x(0) &= \frac{1}{6}t^3 + c_1(0) + c_2 \\ 2 &= c_2, \end{aligned}$$

hence

$$x(t) = \frac{1}{6}t^3 + 3t + 2.$$

2. Separation of Variables. Consider the equation $dy/dx = x/y$.

(a) Use separation of variables to find the general solution.

We have

$$\begin{aligned} y dy &= x dx \\ \int y\, dy &= \int x\, dx + C \\ \frac{1}{2}y^2 &= \frac{1}{2}x^2 + C \\ y^2 &= x^2 + D \\ y &= \pm\sqrt{x^2 + D} \end{aligned}$$

for some constant D.

(b) Find the specific solution with $y(2) = 4$.

Substituting $x = 2$ and $y = 4$ gives

$$\begin{aligned}(4)^2 &= (2)^2 + D \\ 16 &= 4 + D \\ 12 &= D,\end{aligned}$$

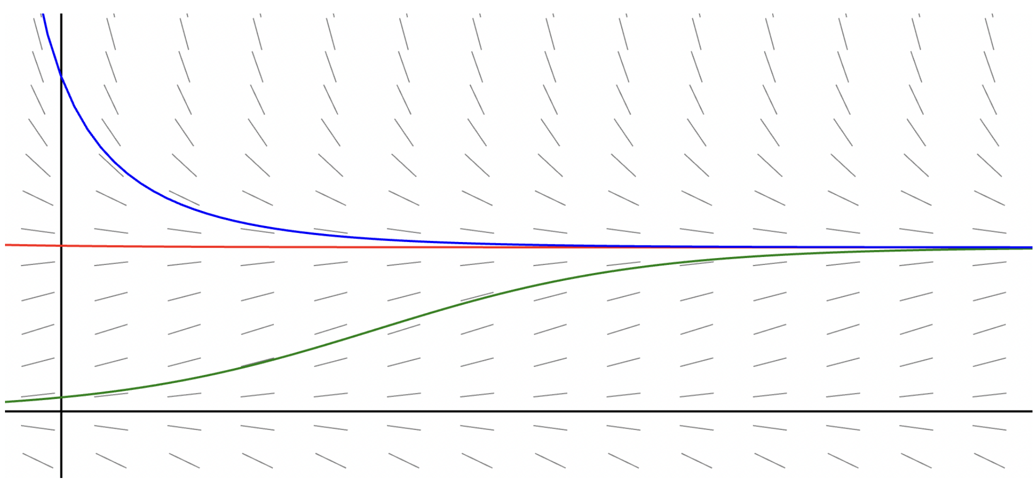
hence

$$y(x) = \pm\sqrt{x^2 + 12}.$$

3. Logistic Growth. The logistic equation $dy/dx = y(1 - y)$ has general solution

$$y(x) = \left[1 + e^{-x}\left(\frac{y(0)}{1 - y(0)}\right)\right]^{-1}.$$

(a) Sketch the slope field of the equation.
(b) Sketch the solutions with initial conditions $y(0) = 0.1$, $y(0) = 1$ and $y(0) = 2$.



4. Damped Oscillations. Consider the equation $x''(t) + 2x'(t) + 2x(t) = 0$.

(a) Find the general solution.

We look for basic solutions of the form $x(t) = e^{\lambda t}$. Substituting gives

$$\begin{aligned}\lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} + 2e^{\lambda t} &= 0 \\ \lambda^2 + 2\lambda + 2 &= 0 \\ \lambda &= \frac{-2 \pm \sqrt{4 - 8}}{2} \\ &= -1 \pm i.\end{aligned}$$

Hence the general solution is

$$\begin{aligned}x(t) &= c_1 e^{(-1+i)t} + c_2 e^{(-1-i)t} \\ &= e^{-t}\left(c_1 e^{it} + c_2 e^{-it}\right)\end{aligned}$$

$$= e^{-t}\left(c_3 \cos t + c_4 \sin t\right)$$

for some constants c_3, c_4.

(b) Find the specific solution with $x(0) = 0$ and $x'(0) = 1$.

Substituting $t = 0$ in $x(t)$ gives

$$\begin{aligned} x(0) &= e^0\left(c_3 1 + c_4 0\right) \\ 0 &= c_3. \end{aligned}$$

Computing $x'(t)$ and substituting $t = 0$ gives

$$\begin{aligned} x'(t) &= -e^{-t}\left(c_3 \cos t + c_4 \sin t\right) + e^{-t}\left(-c_3 \sin t + c_4 \cos t\right) \\ x'(0) &= -e^0\left(c_3 1 + c_4 0\right) + e^0\left(-c_3 0 + c_4 1\right) \\ 1 &= -c_3 + c_4 \\ 1 &= -0 + c_4 \\ 1 &= c_4, \end{aligned}$$

hence

$$x(t) = e^{-t}\left(0 \cos t + 1 \sin t\right) = e^{-t} \sin t.$$

5. Undetermined Coefficients. Consider the equation $x''(t) + x'(t) = t$.

(a) Find the general solution of the homogeneous equation $x''(t) + x'(t) = 0$.

Substituting the guess $x(t) = e^{\lambda t}$ gives

$$\begin{aligned} \lambda^2 e^{\lambda t} + \lambda e^{\lambda t} &= 0 \\ \lambda^2 + \lambda &= 0 \\ \lambda(\lambda + 1) &= 0 \\ \lambda &= 0, -1. \end{aligned}$$

Hence the general solution of $x''(t) + x'(t) = 0$ is

$$x(t) = c_1 e^{0t} + c_2 e^{-1t} = c_1 + c_2 e^{-t}.$$

(b) Find the general solution of the non-homogeneous equation $x''(t) + x'(t) = t$. Use the guess $x_p(t) = A + Bt + Ct^2$ for the particular solution.

To find a particular solution we substitute the guess $x_p(t) = A + Bt + Ct^2$:

$$\begin{aligned} (A + Bt + Ct^2)'' + (A + Bt + Ct^2)' &= t \\ 2C + B + 2Ct &= t \\ (2C)t + (B + 2C) &= 1t + 0. \end{aligned}$$

Comparing coefficients gives $2C = 1$ and $B + 2C = 0$, hence $C = 1/2$ and $B = -2C = -1$. The value of A is arbitrary. Let's just take $A = 0$ to obtain the particular solution

$$x_p(t) = 0 - t + t^2/2.$$

Combining this with the homogeneous solution from part (a) gives the general solution

$$x(t) = x_c(t) + x_p(t) = c_1 + c_2 e^{-t} - t + t^2/2.$$

6. Rules for Laplace Transforms. Use the rules in the attached table to compute the following Laplace transforms.

(a) $\mathscr{L}[t \cdot e^{2t}]$

The table says that $\mathscr{L}[t] = 1/s^2$ and hence

$$\mathscr{L}[t \cdot e^{2t}] = \mathscr{L}[t]_{s \to s-2} = \frac{1}{(s-2)^2}.$$

(b) $\mathscr{L}[e^{2t} \cdot \sin t]$

The table says that $\mathscr{L}[\sin t] = 1/(s^2+1)$ and hence

$$\mathscr{L}[e^{2t} \cdot \sin] = \mathscr{L}[\sin t]_{s \to s-2} = \frac{1}{(s-2)^2+1}.$$

(c) $\mathscr{L}[t \cdot \sin t]$

The table says that $\mathscr{L}[\sin t] = 1/(s^2+1)$ and hence

$$\mathscr{L}[t \cdot \sin] = -\frac{d}{ds}\mathscr{L}[\sin t] = -\frac{d}{ds}\frac{1}{(s^2+1)} = \frac{2s}{(s^2+1)^2}.$$

7. Solve Using Laplace Transforms. Consider the equation $x'(t) + x(t) = 2\sin t$.

(a) Find the partial fraction decomposition of $\frac{2}{(s+1)(s^2+1)}$.

We are looking for A, B, C so that

$$\begin{aligned}
\frac{2}{(s+1)(s^2+1)} &= \frac{A}{s+1} + \frac{Bs+C}{s^2+1} \\
\frac{2}{(s+1)(s^2+1)} &= \frac{A(s^2+1) + (Bs+C)(s+1)}{(s+1)(s^2+1)} \\
2 &= A(s^2+1) + (Bs+C)(s+1) \\
0s^2 + 0s + 2 &= (A+B)s^2 + (B+C)s + (A+C).
\end{aligned}$$

Comparing coefficients gives $A + B = 0$, $B + C = 0$ and $A + C = 2$, which implies that $A = -B = C = 1$. We conclude that

$$\frac{2}{(s+1)(s^2+1)} = \frac{1}{s+1} + \frac{-s+1}{s^2+1} = \frac{1}{s+1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}.$$

(b) Use Laplace transforms to solve the equation with $x(0) = 0$.

Applying Laplace transforms gives

$$\begin{aligned}
x'(t) + x(t) &= 2\sin t \\
sX - x(0) + X &= \frac{2}{s^2+1} \\
(s+1)X &= \frac{2}{s^2+1} \\
X &= \frac{2}{(s+1)(s^2+1)}
\end{aligned}$$

$$
\begin{aligned}
X &= \frac{1}{s+1} - \frac{s}{s^2+1} + \frac{1}{s^2+1} \\
x(t) &= \mathscr{L}^{-1}\left[\frac{1}{s+1}\right] - \mathscr{L}^{-1}\left[\frac{s}{s^2+1}\right] + \mathscr{L}^{-1}\left[\frac{1}{s^2+1}\right] \\
x(t) &= e^{-t} - \cos t + \sin t.
\end{aligned}
$$

8. Discontinuous Input. Consider the step function $H(t)$ and the delta function $\delta(t)$.

(a) Compute the partial fraction decomposition of $\frac{1}{s(s-1)}$.

We are looking for A, B such that

$$
\begin{aligned}
\frac{1}{s(s-1)} &= \frac{A}{s} + \frac{B}{s-1} \\
\frac{1}{s(s-1)} &= \frac{A(s-1)+Bs}{s(s-1)} \\
1 &= A(s-1) + Bs
\end{aligned}
$$

Substituting $s = 0$ gives $1 = A(0-1) = -A$ and substituting $s = 1$ gives $1 = B$, hence

$$\frac{1}{s(s-1)} = -\frac{1}{s} + \frac{1}{s-1}.$$

(b) Use your answer from (a) to evaluate $\mathscr{L}^{-1}\left[\frac{e^{-3s}}{s(s-1)}\right]$

If $F(s) = \mathscr{L}[f(t)]$ then $\mathscr{L}^{-1}\left[e^{-as} \cdot f(t)\right] = H(t-a)f(t-a)$. In our case we have $F(s) = \frac{1}{s(s-1)}$, which implies that

$$
\begin{aligned}
f(t) &= \mathscr{L}^{-1}\left[\frac{1}{s(s-1)}\right] \\
&= \mathscr{L}^{-1}\left[-\frac{1}{s} + \frac{1}{s-1}\right] \\
&= -\mathscr{L}^{-1}\left[\frac{1}{s}\right] + \mathscr{L}^{-1}\left[\frac{1}{s-1}\right] \\
&= -1 + e^t,
\end{aligned}
$$

and hence

$$\mathscr{L}^{-1}\left[\frac{e^{-3s}}{s(s-1)}\right] = H(t-3)f(t-3) = H(t-3)\left(-1 + e^{t-3}\right).$$

(c) Use your answers from (a) and (b) to solve the equation $x''(t) - x'(t) = \delta(t-3)$ with initial conditions $x(0) = x'(0) = 0$.

Applying Laplace transforms gives

$$
\begin{aligned}
x''(t) - x'(t) &= \delta(t-3) \\
s^2X - sx(0) - x'(0) - (sX - x(0)) &= e^{-3s} \\
(s^2 - s)X &= e^{-3s}
\end{aligned}
$$

$$
\begin{aligned}
X &= \frac{e^{-3s}}{s(s-1)} \\
x(t) &= \mathscr{L}^{-1}\left[\frac{e^{-3s}}{s(s-1)}\right] \\
x(t) &= H(t-3)\left(-1+e^{t-3}\right).
\end{aligned}
$$

9. First Order Linear System. Consider the linear system

$$
\begin{cases} x'(t) &= -x(t)+2y(t), \\ y'(t) &= x(t). \end{cases}
$$

(a) Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$.

The eigenvalues are the roots of the characteristic equation:

$$
\begin{aligned}
\begin{vmatrix} -1-\lambda & 2 \\ 1 & 0-\lambda \end{vmatrix} &= 0 \\
(-1-\lambda)(0-\lambda)-(1)(2) &= 0 \\
\lambda^2+\lambda-2 &= 0 \\
(\lambda+2)(\lambda-1) &= 0 \\
\lambda &= 1, -2.
\end{aligned}
$$

The eigenvectors (u, v) for $\lambda = 1$ satisfy

$$
\begin{aligned}
& \begin{pmatrix} -1-1 & 2 \\ 1 & 0-1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\rightsquigarrow \quad & \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\rightsquigarrow \quad & \begin{pmatrix} u \\ v \end{pmatrix} = \text{any multiple of } \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\end{aligned}
$$

The eigenvectors (u, v) for $\lambda = -2$ satisfy

$$
\begin{aligned}
& \begin{pmatrix} -1+2 & 2 \\ 1 & 0+2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\rightsquigarrow \quad & \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\rightsquigarrow \quad & \begin{pmatrix} u \\ v \end{pmatrix} = \text{any multiple of } \begin{pmatrix} 2 \\ -1 \end{pmatrix}
\end{aligned}
$$

(b) Find the general solution of the linear system.

The general solution is

$$
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-2t}.
$$

10. Second Order Linear System. Consider the linear system

$$\begin{cases} x''(t) &=& -3x(t) + 2y(t), \\ y''(t) &=& x(t) - 2y(t). \end{cases}$$

Here are the eigenvalues and eigenvectors of the coefficient matrix:

$$\begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -1^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = -2^2 \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

(a) Use the given information to find the general solution of the system.

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos t + b_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin t + a_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} \cos(2t) + b_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} \sin(2t).$$

(b) Find the specific solution with $x(0) = x'(0) = y(0) = 0$ and $y'(0) = 1$.

Substituting $t = 0$ into $x(t)$ and $y(t)$ gives

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

This implies that $0 = a_1 + 2a_2$ and $0 = a_1 - a_2$, which has solution $a_1 = a_2 = 0$. Now we know that

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = b_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin t + b_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} \sin(2t).$$

Taking derivatives and substituting $t = 0$ gives

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = b_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos t + 2b_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} \cos(2t)$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = b_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2b_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

This implies that $0 = b_1 + 4b_2$ and $1 = b_1 - 2b_2$, which has solution $b_1 = 2/3$ and $b_2 = -1/6$. Hence the solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{2}{3} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin t - \frac{1}{6} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \sin(2t).$$