Math 311 **Exam 2**
Spring 2023 **Fri Apr 28**

No electronic devices are allowed. No collaboration is allowed. There are 5 pages and each page is worth 6 points, for a total of 30 points.

1. Undetermined Coefficients.

(a) Find one solution to the equation $x'(t) + x(t) = e^{2t}$. [Guess: $x_p(t) = Ae^{2t}$.]

Substituting the guess $x_p(t) = Ae^{2t}$ into the equation gives

$$\begin{aligned} x_p'(t) + x_p(t) &= e^{2t} \\ 2Ae^{2t} + Ae^{2t} &= e^{2t} \\ 3Ae^{2t} &= e^{2t} \\ 3A &= 1 \\ A &= 1/3. \end{aligned}$$

Hence $x_p(t) = (1/3)e^{2t}$ is a particular solution.

(b) Find one solution to the equation $x''(t) + x(t) = t^2$. [Guess: $x_p(t) = A + Bt + Ct^2$.]

Substituting the guess $x_p(t) = A + Bt + Ct^2$ into the equation gives

$$\begin{aligned} x_p''(t) + x_p(t) &= t^2 \\ 2C + A + Bt + Ct^2 &= t^2 \\ Ct^2 + Bt + (A + 2C) &= 1t^2 + 0t + 0. \end{aligned}$$

Comparing coefficients gives $C = 1$, $B = 0$ and $A + 2C = 0$, hence $A = -2$. Hence we obtain the particular solution $x_p(t) = t^2 - 2$.

2. Putting it Together. Consider the differential equation $x''(t) + x(t) = t^2$.

(a) Find the general solution of the homogeneous equation $x''(t) + x(t) = 0$.

The basic solutions of a homogeneous linear equation with constant coefficients have the form $x(t) = e^{\lambda t}$. Plugging this in gives an equation for λ:

$$\begin{aligned} \lambda^2 e^{\lambda t} + e^{\lambda t} &= 0 \\ (\lambda^2 + 1)e^{\lambda t} &= 0 \\ \lambda^2 + 1 &= 0 \\ \lambda^2 &= -1 \\ \lambda &= \pm i. \end{aligned}$$

Hence the general solution is $x_c(t) = c_1 e^{it} + c_2 e^{-it}$. By using Euler's formula we can rewrite this as

$$x_c(t) = c_3 \cos t + c_3 \sin t$$

for some real constants c_3, c_4.

(b) Combine 1(b) and 2(a) to find the solution of $x''(t)+x(t) = t^2$ with $x(0) = x'(0) = 1$.

The general solution of $x''(t) + x(t) = t^2$ is the sum of the general homogeneous solution and any one particular solution:

$$x(t) = x_c(t) + x_p(t) = c_3 \cos t + c_4 \sin t + t^2 - 2.$$

The parameters c_3, c_4 are determined by the initial conditions $x(0) = x'(0) = 1$. Substituting $t = 0$ in $x(t)$ gives

$$\begin{aligned} x(0) &= c_3 \cos(0) + c_4 \sin(0) + 0^2 - 2 \\ 1 &= c_3 - 2 \\ c &= 3. \end{aligned}$$

Then substituting $t = 0$ in $x'(t)$ gives

$$\begin{aligned} x'(t) &= -c_3 \sin t + c_4 \cos t + 2t \\ x'(0) &= -c_3 \sin(0) + c_4 \cos(0) + 2(0) \\ 1 &= c_4 \\ c_4 &= 1. \end{aligned}$$

We conclude that

$$x(t) = 3 \cos t + \sin t + t^2 - 2.$$

3. Using Laplace Transforms.

(a) Find the partial fraction decomposition of $\frac{1}{s(s^2+1)}$.

We are looking for A, B, C such that

$$\begin{aligned} \frac{1}{s(s^2+1)} &= \frac{A}{s} + \frac{Bs + C}{s^2 + 1} \\ \frac{1}{s(s^2+1)} &= \frac{A(s^2+1) + (Bs + C)s}{s(s^2+1)} \\ 1 &= A(s^2+1) + (Bs + C)s \\ 0s^2 + 0s + 1 &= (A + B)s^2 + Cs + A. \end{aligned}$$

Comparing coefficients gives $A = 1$, $C = 0$ and $A + B = 0$, hence $B = -1$. We conclude that

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}.$$

(b) Use Laplace transforms to solve the equation $x''(t)+x(t) = 1$ with $x(0) = x'(0) = 0$.

Applying Laplace transforms gives

$$\begin{aligned} s^2 X - sx(0) - x'(0) + X &= 1/s \\ s^2 X + X &= 1/s \\ (s^2+1)X &= 1/s \\ X &= \frac{1}{s(s^2+1)} \\ X &= \frac{1}{s} - \frac{s}{s^2+1} \end{aligned}$$

$$x(t) = \mathscr{L}^{-1}\left[\frac{1}{s}\right] + \mathscr{L}^{-1}\left[\frac{s}{s^2+1}\right]$$
$$x(t) = 1 - \cos t.$$

4. Discontinuous Input. Consider the step function $H(t)$ and the delta function $\delta(t)$.

(a) Solve the following inverse Laplace transform: $\mathscr{L}^{-1}\left[e^{-5s}\cdot\frac{1}{s^2+1}\right]$. Your answer will involve the step function.

Since $\mathscr{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$ we have

$$\mathscr{L}^{-1}\left[e^{-5s}\cdot\frac{1}{s^2+1}\right] = H(t-5)\sin(t-5) = \begin{cases}\sin t & t<5,\\ \sin t + \sin(t-5) & t>5.\end{cases}$$

(b) Solve the equation $x''(t) + x(t) = \delta(t-5)$ with $x(0) = 0$ and $x'(0) = 1$.

Applying Laplace transforms gives

$$\begin{aligned}
s^2X - sx(0) - x'(0) + X &= e^{-5s}\\
s^2X - 1 + X &= e^{-5s}\\
(s^2+1)X &= 1 + e^{-5s}\\
X &= \frac{1}{s^2+1} + e^{-5s}\cdot\frac{1}{s^2+1}\\
x(t) &= \mathscr{L}^{-1}\left[\frac{1}{s^2+1}\right] + \mathscr{L}^{-1}\left[e^{-5s}\cdot\frac{1}{s^2+1}\right]\\
x(t) &= \sin t + H(t-5)\sin(t-5).
\end{aligned}$$

5. Linear Systems. Consider the linear system

$$\begin{cases} x'(t) &= x(t) + y(t),\\ y'(t) &= -2x(t) + 4y(t).\end{cases}$$

(a) Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix}1 & 1\\ -2 & 4\end{pmatrix}$.

The eigenvalues are given by the characteristic equation:

$$\begin{aligned}
\begin{vmatrix}1-\lambda & 1\\ -2 & 4-\lambda\end{vmatrix} &= 0\\
(1-\lambda)(4-\lambda) - (-2)(1) &= 0\\
\lambda^2 - 5\lambda + 4 + 2 &= 0\\
\lambda^2 - 5\lambda + 6 &= 0\\
(\lambda-2)(\lambda-3) &= 0\\
\lambda &= 2, 3.
\end{aligned}$$

The eigenvectors for $\lambda = 2$ are given by

$$\begin{cases} (1-2)u + 1v &= 0, \\ -2u + (4-2)v &= 0, \end{cases}$$
$$\rightsquigarrow \quad \begin{cases} -1u + 1v &= 0, \\ -2u + 2v &= 0, \end{cases}$$
$$\rightsquigarrow \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The eigenvectors for $\lambda = 3$ are given by

$$\begin{cases} (1-3)u + 1v &= 0, \\ -2u + (4-3)v &= 0, \end{cases}$$
$$\rightsquigarrow \quad \begin{cases} -2u + 1v &= 0, \\ -2u + 1v &= 0, \end{cases}$$
$$\rightsquigarrow \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

In summary, we have found that

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(b) Find the general solution of the linear system.

The general solution of the linear system is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}.$$