Math 311 — Spring 2023
Practice for Exam 2 — Drew Armstrong

A *linear operator* L takes functions to functions and satisfies

$$L[ay_1(x) + by_2(x)] = aL[y_1(x)] + bL[y_2(x)].$$

In this section we looked at *non-homogeneous linear equations* $L[y(x)] = f(x)$. The general solution is $y(x) = y_c(x)+y_p(x)$ where $y_c(x)$ is the general solution of the homogeneous equation $L[y(x)] = 0$ and $y_p(x)$ is any one particular solution of the non-homogeneous equation.

1. Undetermined Coefficients. In each case find one particular solution:

(a) $x'(t) + x(t) = 5$ [Hint: Guess $x_p(t) = A$.]
(b) $x'(t) + x(t) = \sin(2t)$ [Hint: Guess $x_p(t) = A\cos(2t) + B\sin(2t)$.]
(c) $x'(t) + x(t) = t^2$ [Hint: Guess $x_p(t) = A + Bt + Ct^2$.]
(d) $x''(t) + x(t) = e^{2t}$ [Hint: Guess $x_p(t) = Ae^{2t}$.]

2. Putting It Together. Solve the initial value problems:

(a) $x'(t) + x(t) = 5$; $x(0) = 1$
(b) $x'(t) + x(t) = \sin(2t)$; $x(0) = 1$
(c) $x'(t) + x(t) = t^2$; $x(0) = 1$
(d) $x''(t) + x(t) = e^{2t}$; $x(0) = 1$, $x'(0) = 0$

3. Partial Fractions. Find the partial fraction decomposition of the following expressions:

(a) $\frac{1}{s(s+1)}$
(b) $\frac{1}{s(s-1)(s-2)}$
(c) $\frac{1}{s^2(s^2+1)}$

4. Laplace Transform Rules. Use the information in the table to compute the following Laplace transforms:

(a) $\mathscr{L}[t \cdot e^{2t}]$
(b) $\mathscr{L}[t^2 \cdot e^{2t}]$
(c) $\mathscr{L}[e^{3t} \cdot \cos(2t)]$
(d) $\mathscr{L}[t \cdot \cos t]$
(e) $\mathscr{L}^{-1}\left[\frac{1}{(s-3)^2+4}\right]$
(f) $\mathscr{L}^{-1}[e^{-4s}/s^3]$

5. Using Laplace Transforms. Use Laplace transforms to solve the following initial value problems:

(a) $x'(t) + x(t) = 2$; $x(0) = 0$
(b) $x''(t) - x'(t) = e^{2t}$; $x(0) = x'(0) = 0$
(c) $x'(t) + x(t) = \delta(t-1)$; $x(0) = 1$
(d) $x''(t) + x(t) = t$; $x(0) = x'(0) = 1$
(e) $x''(t) + x(t) = \delta(t-4)$; $x(0) = x'(0) = 0$

6. Eigenvectors. Find the eigenvalues and eigenvectors of the following matrices:

(a) $\begin{pmatrix} 5 & -4 \\ 6 & -5 \end{pmatrix}$

(b) $\begin{pmatrix} 7 & -16 \\ 8 & -17 \end{pmatrix}$

7. First Order Linear Systems. Consider the first order linear system:

$$\begin{cases} x'(t) &= 5x(t) - 4y(t), \\ y'(t) &= 6x(t) - 5y(t). \end{cases}$$

(a) Use your answer from 6(a) to find the general solution.
(b) Find the specific solution with $x(0) = 1$ and $y(0) = 2$.

8. Second Order Linear Systems. Consider the second order linear system:

$$\begin{cases} x''(t) &= 7x(t) - 16y(t), \\ y''(t) &= 8x(t) - 17y(t). \end{cases}$$

(a) Use your answer from 6(b) to find the general solution.
(b) Find the particular solution with $x(0) = x'(0) = y(0) = 0$ and $y'(0) = 1$.

Rules for Laplace Transforms. Let $\mathscr{L}[f(t)] = F(s)$. Then

- $\mathscr{L}[tf(t)] = -F'(s)$
- $\mathscr{L}[e^{at}f(t) = F(s-a)$
- $\mathscr{L}[f'(t)] = sF(s) - f(0)$
- $\mathscr{L}[f''(t)] = s^2F(s) - sf(0) - f'(0)$
- $\mathscr{L}[H(t-a)f(t-a)] = e^{-as}F(s)$, where $H(t-a) = \begin{cases} 0 & t < a, \\ 1 & t > a. \end{cases}$

Table of Laplace Transforms.

- $\mathscr{L}[0] = 0$
- $\mathscr{L}[1] = 1/s$
- $\mathscr{L}[e^{as}] = 1/(s-a)$
- $\mathscr{L}[t] = 1/s^2$
- $\mathscr{L}[t^n] = n!/s^{n+1}$
- $\mathscr{L}[\cos(kt)] = s/(s^2+k^2)$
- $\mathscr{L}[\sin(kt)] = k/(s^2+k^2)$
- $\mathscr{L}[\delta(t)] = 1$, where $\delta(t)$ is the Dirac delta function
- $\mathscr{L}[\delta(t-a)] = e^{-as}$
- $\mathscr{L}[H(t-a)] = e^{-as}/s$