
Problem 1. Describing Lines in $\mathbb{R}^2$ and $\mathbb{R}^3$. In each part, there are infinitely many correct ways to express the answer.

(a) Find a parametrization for the line in $\mathbb{R}^2$ passing through points $(3,0)$ and $(1,1)$.
(b) Eliminate the parameter to find the equation of the line from part (a).
(c) Find a parametrization for the line in $\mathbb{R}^3$ passing through points $(1,-1,1)$ and $(0,3,3)$.
(d) Eliminate the parameter to find the **equations of two planes** in $\mathbb{R}^3$ whose intersection is the line from part (c).

Problem 2. A Plane in $\mathbb{R}^3$. The following three points in $\mathbb{R}^3$ determine a plane:

$$P = (1,0,0), \quad Q = (1,-1,0), \quad R = (1,2,3).$$

(a) Find a vector that is perpendicular to this plane. [Hint: Use the cross product.]
(b) Use your answer from part (a) to find the equation of the plane. [Recall: The plane in $\mathbb{R}^3$ that passes through the point (x_0, y_0, z_0) and is perpendicular to the vector $\langle a,b,c\rangle$ has the equation $\langle a,b,c\rangle \bullet \langle x-x_0, y-y_0, z-z_0\rangle = 0$.]

Problem 3. The Intersection of Two Planes in $\mathbb{R}^3$. The following system of two linear equations in three unknowns represents the intersection of two planes in $\mathbb{R}^3$:

$$\begin{cases} 2x + y - z = 3, \\ x - y + 2z = 1. \end{cases}$$

(a) Express the intersection as a parametrized line in $\mathbb{R}^3$. [Hint: There are many ways to express the answer. I think the easiest way is to let $t = z$ be the parameter. Then solve for x and y in terms of t.]
(b) The two planes have normal vectors $\mathbf{u} = \langle 2,1,-1\rangle$ and $\mathbf{v} = \langle 1,-1,2\rangle$. Compute the cross product $\mathbf{u}\times\mathbf{v}$. How is this related to your answer from part (a)?

Problem 4. Projectile Motion. A projectile of mass m is launched from the point $(0,0)$ with an initial speed of v, at an angle of θ above the horizontal. Let $\mathbf{r}(t) = (x(t), y(t))$ be the position of the particle at time t.

(a) Neglecting air resistance, Galileo tells us that the acceleration is $\mathbf{r}''(t) = \langle 0, -g\rangle$ for some constant $g > 0$.[1] Use this to find the position $\mathbf{r}(t)$. [Hint: Integrate $\mathbf{r}''(t)$ twice, using the initial conditions $\mathbf{r}(0) = (0,0)$ and $\mathbf{r}'(0) = \langle v\cos\theta, v\sin\theta\rangle$.]
(b) Show that the projectile travels horizontal distance $v^2\sin(2\theta)/g$ before hitting the ground. [Hint: Set $y(t) = 0$ and solve for t.]
(c) Find the value of θ that maximizes the horizontal distance traveled. [Hint: Think of the distance as a function of θ. Compute $d/d\theta$.]

Problem 5. Derivatives of Dot Products and Cross Products. Let $\mathbf{f},\mathbf{g} : \mathbb{R}\to\mathbb{R}^3$ be any[2] two parametrized paths in $\mathbb{R}^3$. Then we have the following "product rules":

$$[\mathbf{f}(t)\bullet\mathbf{g}(t)]' = \mathbf{f}(t)\bullet\mathbf{g}'(t) + \mathbf{f}'(t)\bullet\mathbf{g}(t),$$
$$[\mathbf{f}(t)\times\mathbf{g}(t)]' = \mathbf{f}(t)\times\mathbf{g}'(t) + \mathbf{f}'(t)\times\mathbf{g}(t).$$

[1] In particular, the acceleration does not depend on the mass m.
[2] We assume that the derivatives $\mathbf{f}'(t)$ and $\mathbf{g}'(t)$ exist.

(a) Let $\mathbf{r}(t)$ be the trajectory of a particle traveling on the surface of a sphere centered at $(0, 0, 0)$. In this case, prove that $\mathbf{r}(t) \bullet \mathbf{r}'(t) = 0$ for all t. [Hint: By assumption we have $\|\mathbf{r}(t)\| = c$ for some constant c independent of t. Use the fact that $\|\mathbf{r}(t)\|^2 = \mathbf{r}(t) \bullet \mathbf{r}(t)$.]

(b) Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ be the trajectory of a particle in space, and suppose that we have $\mathbf{r}''(t) = c(t)\mathbf{r}(t)$ for some scalar function $c : \mathbb{R} \to \mathbb{R}$. In this case, prove that

$$[\mathbf{r}(t) \times \mathbf{r}'(t)]' = \langle 0, 0, 0 \rangle \text{ for all } t.$$

[Hint: Recall that we have $\mathbf{v} \times \mathbf{v} = \langle 0, 0, 0 \rangle$ for any vector $\mathbf{v} \in \mathbb{R}^3$.]

Problem 6. Universal Gravitation. Choose a coordinate system with a star at $(0, 0, 0)$ and let $\mathbf{r}(t)$ be the position of a planet at time t. Newton tells us that the planet feels a gravitational force $\mathbf{F}(t)$ pointed directly toward the sun. Specifically, we have

$$\mathbf{F}(t) = -\frac{GMm}{\|\mathbf{r}(t)\|^3}\mathbf{r}(t),$$

where M is the mass of the star, m is the mass of the planet and G is a universal constant.

(a) Use Newton's Second Law, $\mathbf{F}(t) = m\mathbf{r}''(t)$, to show that $\mathbf{r}''(t) = (-GM/\|\mathbf{r}(t)\|^3)\mathbf{r}(t)$.

(b) The vector $\mathbf{h}(t) := \mathbf{r}(t) \times \mathbf{r}'(t)$ is called *angular momentum*. Use part (a) and Problem 5(b) to prove that $\mathbf{h}'(t) = \langle 0, 0, 0 \rangle$ for all t. It follows that $\mathbf{h}(t)$ is a constant vector.

(c) Let's say that $\mathbf{h} = \langle h_1, h_2, h_3 \rangle$ is the constant angular momentum vector of two-body system. Explain why the planet always stays within the plane $h_1x + h_2y + h_3z = 0$. This plane is called the *ecliptic*.