Math 310 **Spring 2024**
Homework 1 **Drew Armstrong**

Problem 1. Lines and Circles. The parametrized curve in part (a) is a line. The parametrized curve in part (b) is a circle. In each case, compute the velocity vector $\mathbf{f}'(t) = \langle x'(t), y'(t)\rangle$ and speed $\|\mathbf{f}'(t)\| = \sqrt{x'(t)^2 + y'(t)^2}$ at time t. Also eliminate t to find an equation for the curve in terms of x and y. [Hint: In part (b) look at $(x-a)^2 + (y-b)^2$.]

(a) $\mathbf{f}(t) = (x(t), y(t)) = (p + ut, q + vt)$ where p, q, u, v are constants.
(b) $\mathbf{f}(t) = (x(t), y(t)) = (a + r\cos(\omega t), b + r\sin(\omega t))$ where a, b, r, ω are constants.

Problem 2. An Interesting Parametrized Curve. Consider the parametrized curve

$$\mathbf{f}(t) = (x(t), y(t)) = (t^2 - 1, t^3 - t).$$

(a) Compute the velocity vector $\mathbf{f}'(t) = \langle x'(t), y'(t)\rangle$ at time t.
(b) Find the slope of the tangent line at time t. [Hint: $dy/dx = (dy/dt)/(dx/dt)$.]
(c) Find all points on the curve where the tangent is horizontal or vertical.
(d) Sketch the curve. [Hint: Plot several points. Use a computer if you want.]
(e) Eliminate t to find an equation relating x and y. [Hint: This kind of problem is impossible in general, but in this case there is a very nice answer. Since $x = t^2 - 1$ we have $t = \pm\sqrt{x+1}$. Substitute this into y and simplify as much as possible.]

Problem 3. Arc Length. Consider the parametrized curve $\mathbf{f}(t) = (t^2, t^3)$. Find the arc length of this curve between times $t = 0$ and $t = 1$. [Hint: The arc length is the integral of the speed: $\int_0^1 \|\mathbf{f}'(t)\|\,dt$. Arc length is generally impossible to compute by hand but in this case there is a lucky accident that allows the integral to be computed via substitution.]

Problem 4. A Triangle in the Plane. Consider the following points in $\mathbb{R}^2$:

$$P = (-2, 1), \quad Q = (1, -2), \quad R = (2, 3).$$

(a) Draw the three points P, Q, R, the midpoints $(P+Q)/2$, $(P+R)/2$, $(Q+R)/2$ and the centroid $(P+Q+R)/3$.
(b) Find the coordinates of the three side vectors $\mathbf{u} = \vec{PQ}, \mathbf{v} = \vec{PR}, \mathbf{w} = \vec{QR}$. Check that $\mathbf{u} + \mathbf{w} = \mathbf{v}$. This is true because of the rule $\vec{PQ} + \vec{QR} = \vec{PR}$.
(c) Use the length formula to compute the three side lengths $\|\mathbf{u}\|, \|\mathbf{v}\|, \|\mathbf{w}\|$.
(d) Use the dot product to compute the three angles of the triangle. After computing the angles, check that they sum to $180°$. [Hint: Let α, β, γ be the angles at P, Q, R. The dot product theorem says that $\cos\alpha = \mathbf{u} \bullet \mathbf{v}/(\|\mathbf{u}\|\|\mathbf{v}\|)$. What about β and γ?]

Problem 5. Some Properties of Vector Arithmetic. Consider three vectors in $\mathbb{R}^3$:

$$\mathbf{u} = \langle u_1, u_2, u_3\rangle, \quad \mathbf{v} = \langle v_1, v_2, v_3\rangle, \quad \mathbf{w} = \langle w_1, w_2, w_3\rangle.$$

(a) For any real number $a \in \mathbb{R}$ check that $(a\mathbf{u}) \bullet \mathbf{v} = \mathbf{u} \bullet (a\mathbf{v}) = a(\mathbf{u} \bullet \mathbf{v})$.
(b) Check the distributive property: $(\mathbf{u} + a\mathbf{v}) \bullet \mathbf{w} = \mathbf{u} \bullet \mathbf{w} + a(\mathbf{v} \bullet \mathbf{w})$.
(c) Substitute $\mathbf{w} = \mathbf{u} + a\mathbf{v}$ in part (b) to show that

$$(\mathbf{u} + a\mathbf{v}) \bullet (\mathbf{u} + a\mathbf{v}) = \mathbf{u} \bullet \mathbf{u} + a^2(\mathbf{v} \bullet \mathbf{v}) + 2a(\mathbf{u} \bullet \mathbf{v})$$

(d) Substitute $a = -1$ in part (c) to show that

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u} \bullet \mathbf{v}).$$

[Hint: Recall that $\|\mathbf{x}\|^2 = \mathbf{x} \bullet \mathbf{x}$ for any vector $\mathbf{x}$.]