

No electronic devices are allowed. No collaboration is allowed. There are 5 pages and each page is worth 6 points, for a total of 30 points.

1. Integrating a Scalar Over a Region in the Plane.

- (a) Integrate $f(x, y) = xy$ over the rectangle with $0 \leq x \leq 2$ and $0 \leq y \leq 1$.

$$\begin{aligned} \int_0^1 \left(\int_0^2 xy \, dx \right) dy &= \int_0^1 \left[\frac{1}{2} x^2 y \right]_0^2 dy \\ &= \int_0^1 2y \, dy \\ &= [y^2]_0^1 \\ &= 1. \end{aligned}$$

- (b) Integrate $f(x, y) = x^2 + y^2$ over the unit circle $x^2 + y^2 \leq 1$. [Hint: Use polar coordinates.]

The unit circle is parametrized by $x = r \cos \theta$ and $y = r \sin \theta$ with $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Since $x^2 + y^2 = r^2$ and $dx dy = r \, dr d\theta$ we have

$$\begin{aligned} \iint_{\text{circle}} (x^2 + y^2) \, dx dy &= \iint r^2 r \, dr d\theta \\ &= \int_0^{2\pi} \cdot \int_0^1 r^3 \, dr \\ &= 2\pi \left[\frac{1}{4} r^4 \right]_0^1 \\ &= \pi/2. \end{aligned}$$

2. Cylindrical and Spherical Coordinates.

- (a) Use cylindrical coordinates to integrate the function $f(x, y, z) = z$ over the cylinder defined by $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 1$.

Since $dx dy dz = r \, dr d\theta dz$ we have

$$\begin{aligned} \iiint_{\text{cylinder}} z \, dx dy dz &= \iiint z r \, dr d\theta dz \\ &= \int_0^{2\pi} d\theta \cdot \int_0^1 r \, dr \cdot \int_0^1 z \, dz \\ &= 2\pi \cdot \left[\frac{1}{2} r^2 \right]_0^1 \cdot \left[\frac{1}{2} z^2 \right]_0^1 \\ &= \pi/2. \end{aligned}$$

- (b) Use spherical coordinates to compute the **volume of a sphere of radius 1**. [Hint: You can take $0 \leq \rho \leq 1$, $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$.]

Since $dx dy dz = \rho^2 \sin \phi d\rho d\theta d\phi$ we have

$$\begin{aligned}
 \text{Volume} &= \iiint_{\text{sphere}} 1 dx dy dz \\
 &= \iiint \rho^2 \sin \phi d\rho d\theta d\phi \\
 &= \int_0^{2\pi} d\theta \cdot \int_0^1 \rho^2 d\rho \cdot \int_0^\pi \sin \phi d\phi \\
 &= 2\pi \cdot \left[\frac{1}{3} \rho^3 \right]_0^1 \cdot [-\cos \phi]_0^\pi \\
 &= 2\pi \cdot \frac{1}{3} \cdot [-(-1) + 1] \\
 &= 4\pi/3.
 \end{aligned}$$

3. Surface Area. Consider the following parametrized surface in 3D:

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, v \rangle \quad \text{with } 0 \leq u \leq 1 \text{ and } 0 \leq v \leq 2\pi.$$

- (a) Compute the tangent vectors \mathbf{r}_u and \mathbf{r}_v , and the normal vector $\mathbf{r}_u \times \mathbf{r}_v$.

We have

$$\begin{aligned}
 \mathbf{r}_u &= \langle \cos v, \sin v, 0 \rangle, \\
 \mathbf{r}_v &= \langle -u \sin v, u \cos v, 1 \rangle,
 \end{aligned}$$

and hence

$$\begin{aligned}
 \mathbf{r}_u \times \mathbf{r}_v &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 1 \end{pmatrix} \\
 &= \langle \sin v, -\cos v, u \cos^2 v + u \sin^2 v \rangle \\
 &= \langle \sin v, -\cos v, u \rangle.
 \end{aligned}$$

- (b) Use your answer from part (a) to set up an integral to compute the **area of the surface** and simplify as much as possible. [This integral is too difficult to evaluate by hand; at least, for me it is.]

The surface area is

$$\begin{aligned}
 \iint_{\text{surface}} 1 dA &= \iint 1 \|\mathbf{r}_u \times \mathbf{r}_v\| du dv \\
 &= \iint \sqrt{\sin^2 v + \cos^2 v + u^2} du dv \\
 &= \iint \sqrt{1 + u^2} du dv \\
 &= \int_0^{2\pi} dv \cdot \int_0^1 \sqrt{1 + u^2} du
 \end{aligned}$$

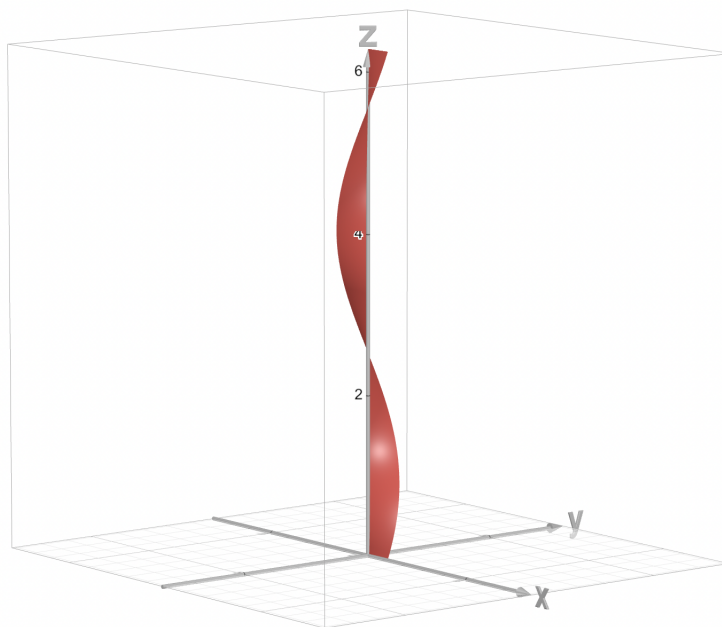
$$= 2\pi \cdot \int_0^1 \sqrt{1+u^2} du.$$

This last integral is not easy to compute. My computer says that

$$\int_0^1 \sqrt{1+u^2} du = \frac{1}{2} \left(\sqrt{2} + \ln(1 + \sqrt{2}) \right),$$

so the area of the surface is $\pi (\sqrt{2} + \ln(1 + \sqrt{2})) \approx 7.2$.

Remark: This surface is called a *helicoid*. It looks like a twisted ribbon of width 1 and length 2π :



The untwisted ribbon would have surface area $2\pi \approx 6.8$. The twisted ribbon has been stretched so its area is slightly larger.

4. Conservative Vector Fields. Consider the vector field $\mathbf{F}(x, y, z) = \langle y, x + z, y \rangle$.

- (a) Compute the curl $\nabla \times \mathbf{F}$ in order to verify that the field \mathbf{F} is conservative. You need to show the steps of the computation, not just the final answer.

Let $\mathbf{F} = \langle P, Q, R \rangle = \langle y, x + z, y \rangle$. The curl is

$$\begin{aligned} \nabla \times \mathbf{F} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{pmatrix} \\ &= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \\ &= \langle (y)_y - (x + z)_z, (y)_z - (y)_x, (x + y)_x - (y)_y \rangle \\ &= \langle 1 - 1, 0 - 0, 1 - 1 \rangle \\ &= \langle 0, 0, 0 \rangle, \end{aligned}$$

which implies that \mathbf{F} is conservative.

- (b) Find a specific scalar function $f(x, y, z)$ such that $\nabla f(x, y, z) = \mathbf{F}(x, y, z)$. [Hint: Integrate \mathbf{F} along any path in the plane that ends at the fixed point (x, y, z) .]

The simplest choice is $\mathbf{r}(t) = \langle tx, ty, tz \rangle$ with $0 \leq t \leq 1$. Then we can take

$$\begin{aligned} f(x, y, z) &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) dt \\ &= \int_0^1 \mathbf{F}(tx, ty, tz) \bullet \langle x, y, z \rangle dt \\ &= \int_0^1 \langle ty, tx + tz, ty \rangle \bullet \langle x, y, z \rangle dt \\ &= \int_0^1 ((ty)x + (tx + tz)y + (ty)z) dz \\ &= (yx + xy + zy + yz) \int_0^1 t dt \\ &= 2(xy + yz) \left[\frac{1}{2} t^2 \right]_0^1 \\ &= xy + yz. \end{aligned}$$

Check:

$$\nabla(xy + yz) = \langle (xy + yz)_x, (xy + yz)_y, (xy + yz)_z \rangle = \langle y, x + z, y \rangle.$$

5. Green's Theorem. Consider the vector field $\mathbf{F}(x, y) = \langle -y + e^x, x + e^y \rangle$ in the plane.

- (a) Compute the scalar function $\text{Curl}(\mathbf{F})$.

Let $\mathbf{F} = \langle P, Q \rangle = \langle -y + e^x, x + e^y \rangle$. Then

$$\begin{aligned} \text{Curl}(\mathbf{F}) &= Q_x - P_y \\ &= (x + e^y)_x - (-y + e^x)_y \\ &= 1 - (-1) \\ &= 2. \end{aligned}$$

- (b) Compute the integral of $\text{Curl}(\mathbf{F})$ over the unit disk $x^2 + y^2 \leq 1$.

This can be solved using polar coordinates, but I will use the fact that I already know the area of the unit disk:

$$\iint_{\text{disk}} \text{Curl}(\mathbf{F}) dx dy = \iint_{\text{disk}} 2 dx dy = 2 \int_{\text{disk}} 1 dx dy = 2\pi.$$

- (c) Compute the integral of \mathbf{F} around the curve $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \leq t \leq 2\pi$. [Hint: This integral is too difficult to compute directly, but there is a shortcut.]

If D is the unit disk then $\mathbf{r}(t)$ is the boundary ∂D . Combining Green's Theorem and part (b) gives

$$\int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) dt = \int_{\partial D} \mathbf{F} \bullet \mathbf{T} = \iint_D \text{Curl}(\mathbf{F}) dx dy = 2\pi.$$

Remark: Actually, the integral in part (c) **isn't** too difficult to compute directly. (A few students were able to do it.) It turns out that the chain rule is all we need:

$$\begin{aligned}
 \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) dt &= \int_0^{2\pi} \mathbf{F}(\cos t, \sin t) \bullet \langle -\sin t, \cos t \rangle dt \\
 &= \int_0^{2\pi} \langle -\sin t + e^{\cos t}, \cos t + e^{\sin t} \rangle \bullet \langle -\sin t, \cos t \rangle dt \\
 &= \int_0^{2\pi} (\sin^2 t - e^{\cos t} \sin t + \cos^2 t + e^{\sin t} \cos t) dt \\
 &= \int_0^{2\pi} (1 - e^{\cos t} \sin t + e^{\sin t} \cos t) dt \\
 &= [t + e^{\cos t} + e^{\sin t}]_0^{2\pi} \\
 &= 2\pi.
 \end{aligned}$$

More generally, we can take $\mathbf{F}(x, y) = \langle -y + f(x), x + g(y) \rangle$ for **any** functions $f(x)$ and $g(y)$ with antiderivatives $F'(x) = f(x)$ and $G'(y) = g(y)$. Then we still have $\text{Curl}(\mathbf{F}) = 2$, so part (b) is still 2π . And the explicit computation of part (c) becomes

$$\begin{aligned}
 \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) dt &= \int_0^{2\pi} \mathbf{F}(\cos t, \sin t) \bullet \langle -\sin t, \cos t \rangle dt \\
 &= \int_0^{2\pi} \langle -\sin t + f(\cos t), \cos t + g(\sin t) \rangle \bullet \langle -\sin t, \cos t \rangle dt \\
 &= \int_0^{2\pi} (\sin^2 t - f(\cos t) \sin t + \cos^2 t + g(\sin t) \cos t) dt \\
 &= \int_0^{2\pi} (1 - f(\cos t) \sin t + g(\sin t) \cos t) dt \\
 &= [t + F(\cos t) + G(\sin t)]_0^{2\pi} \\
 &= 2\pi.
 \end{aligned}$$