HW 5 due Tues

Quiz 5 on Wed

Final Project due next Fri June 24.

Now: Chapter 6 (Vector Calculus)

Recall: Given vector field

$$\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$$

and a curve $\vec{r} : \mathbb{R} \to \mathbb{R}^n$, we define the "line integral"

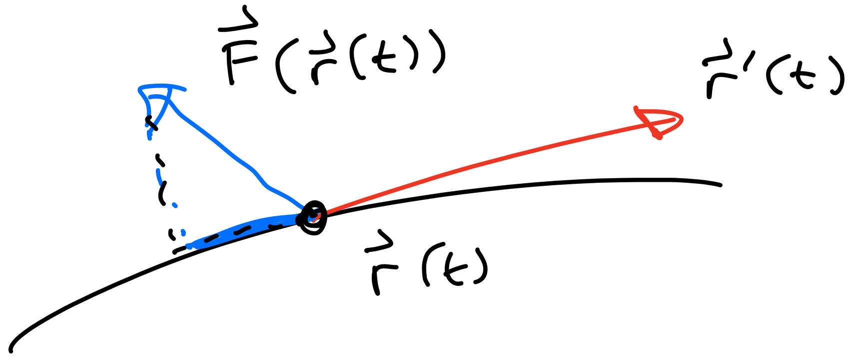
$$\int_{\text{curve}} \vec{F} = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

= sum the component of $\vec{F}$ along the curve

= "on average, how much does $\vec{F}$ point in the direction of the curve?"

$= 0$ if $\vec{F} \perp$ curve at every point

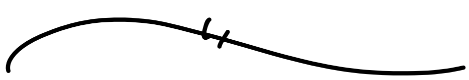
< 0 if $\vec{F}$ points against the curve.



here $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) < 0$

Physics: $\vec{F}$ force field.

$\int_{\text{curve}} \vec{F}$ = amount of KE added to particle by the field. ("speed")



Fund Thm Line Integrals:

If $\vec{F} = \nabla f$ then

$$\int_{\text{curve}} \vec{F} = f(\text{end point}) - f(\text{start point})$$

Proof:

$$\int_{\text{curve}} \vec{F} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

CHAIN RULE

$$= \int_a^b \frac{d}{dt}\left[f(\vec{r}(t))\right] dt$$

Calc I

$$= f(\vec{r}(b)) - f(\vec{r}(a)) \quad \checkmark$$

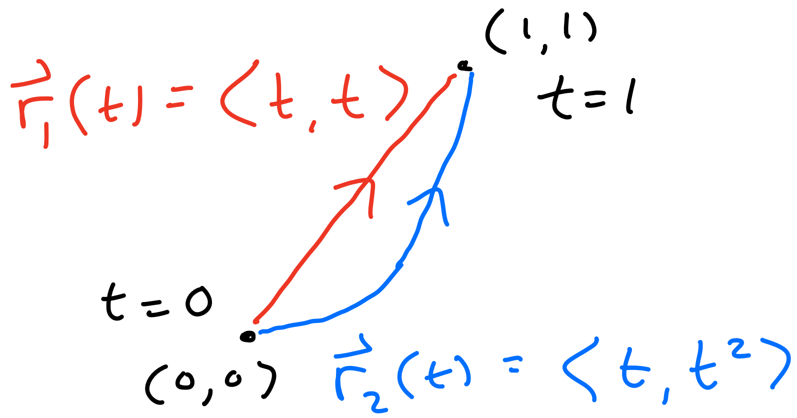
Consequence: If $\vec{F} = \nabla f$ then

$\int_{\text{curve}} \vec{F}$ only depends on the endpoints, not on the shape of the curve.

Example:

$$\vec{F} = \nabla(xy + y)$$

$$= \langle y, x+1 \rangle$$



$$\int_0^1 \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1(t)\, dt$$

$$= \int \langle t, t+1 \rangle \cdot \langle 1, 1 \rangle \, dt$$

$$= \int (t + (t+1))\, dt$$

$$= \int (2t+1)\, dt$$

$$= \left[2 \cdot \frac{t^2}{2} + t \right]_0^1$$

$$= 1 + 1 = 2.$$

$$\int_0^1 \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t)\, dt$$

$$= \int \langle t^2, t+1 \rangle \cdot \langle 1, 2t \rangle \, dt$$

$$= \int \left[t^2 + (t+1)(2t) \right] dt$$

$$= \int \left(t^2 + 2t^2 + 2t \right) dt$$

$$= \int \left(3t^2 + 2t \right) dt$$

$$= \left[3 \cdot \frac{t^3}{3} + 2 \cdot \frac{t^2}{2} \right]_0^1$$

$$= 1 + 1 = 2.$$ SAME ✓

In fact:

$$\int_{\text{curve}} \vec{F} = f(\text{end point}) - f(\text{start})$$

$$= f(1,1) - f(0,0)$$

$$= (1\cdot 1 + 1) - (0\cdot 0 + 0)$$

$$= 2.$$

That's why the two paths give the same answer.

Now let's change $\vec{F}$ a little bit

$$\vec{F}(x,y) = \langle y, x+1 \rangle$$

$$\vec{G}(x,y) = \langle y, 2x+1 \rangle$$

Integrate $\vec{G}$ along the two paths.

$$\int_0^1 \vec{G}(\underbrace{\cancel{\vec{r}_1(t)}}_{t,\,t}) \cdot \underbrace{\cancel{\vec{r}_1{}'(t)}}_{1,\,1}\, dt$$

$$= \int \langle t, 2t+1 \rangle \cdot \langle 1, 1 \rangle \, dt$$

$$= \int (t + (2t+1))\, dt$$

$$= \int (3t+1)\, dt$$

$$= \left(3 \cdot \frac{t^2}{2} + t\right)\Big|_0^1$$

$$= \frac{3}{2} + 1 = \frac{5}{2}.$$

$$\int_0^1 \vec{G}(\cancel{\vec{r}_2(t)}) \circ \vec{r}_2{}'(t)\,dt$$

$$\langle t, t^2 \rangle \qquad \langle 1, 2t \rangle$$

$$= \int \langle t^2, 2t+1 \rangle \cdot \langle 1, 2t \rangle\,dt$$

$$= \int \left(t^2 + (2t+1)(2t)\right) dt$$

$$= \int \left(t^2 + 4t^2 + 2t\right) dt$$

$$= \int \left(5t^2 + 2t\right) dt$$

$$= \left[5 \cdot \frac{t^3}{3} + 2 \cdot \frac{t^2}{2}\right]_0^1$$

$$= \frac{5}{3} + 1 = \frac{8}{3} \neq \frac{5}{2}$$

NOT THE SAME !

Today we'll discuss what went wrong.

But first, Kinetic Energy.

Consider a moving particle $\vec{r}(t)$ with mass m. Define

$$KE(t) = \underbrace{\frac{1}{2} m \| \vec{r}\,'(t) \|^2}_{\text{WHY ?}}$$

Suppose force field $\vec{F}$ acts on the particle, so $\vec{F}(\vec{r}(t)) = m\vec{r}\,''(t)$.

Compute $KE'(t)$.

$$KE(t) = \frac{1}{2} m \| \vec{r}\,'(t) \|^2$$

$$= \frac{1}{2} m \underbrace{\vec{r}\,'(t) \cdot \vec{r}\,'(t)}$$

Product Rule

$$KE'(t) = \frac{1}{2}m\left[\vec{r}''(t)\cdot\vec{r}'(t) + \vec{r}'(t)\cdot\vec{r}''(t)\right]$$

$$= \frac{1}{2}m\left[2\vec{r}''(t)\cdot\vec{r}'(t)\right]$$

$$= \underbrace{m\vec{r}''(t)}\cdot\vec{r}'(t)$$

$$= \underbrace{\vec{F}(\vec{r}(t))\cdot\vec{r}'(t)}_{\text{what do we see?}}.$$

KE'(t) looks familiar!

$$KE(t) = \int \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt$$

$$\int_{\text{curve}} \vec{\text{Force}} = KE(\text{end}) - KE(\text{start})$$

$$= \text{increase in } KE$$

Applies for ANY force field.

Now, assume $\vec{F}$ is conservative:

$$\vec{F} = -\nabla f \text{ for some } f.$$

Then we also have

$$\int_{\text{curve}} \vec{F} = \int -\nabla f$$

$$= -\int \nabla f$$

$$= -\left[f(\text{end}) - f(\text{start}) \right]$$

$$= f(\text{start}) - f(\text{end})$$

↑

Fund Thm Line Integrals

So let's define the potential energy

$$PE(t) = f(\vec{r}(t)).$$

Then combining the above equations:

$$KE(\text{end}) - KE(\text{start})$$

$$= PE(\text{start}) - PE(\text{end}).$$

$$KE(\text{start}) + PE(\text{start})$$

$$= KE(\text{end}) + PE(\text{end}).$$

"Conservation of Mechanical Energy"

Energy is converted between KE & PE but never destroyed.

This is <u>why</u> gradient vector fields are called "conservative".

Example: Gravity near planet.

$$\vec{F}(x,y,z) = \langle 0, 0, -mg \rangle$$

$$\vec{r}(0) = \langle 0,0,0 \rangle$$

$$\vec{r}\,'(0) = \langle 0,0,v \rangle \quad \text{up. } (v>0)$$

$$m\vec{r}\,''(t) = \vec{F}(\vec{r}(t))$$

$$m\vec{r}\,''(t) = \langle 0,0,-mg \rangle$$

$$\vec{r}\,''(t) = \langle 0,0,-g \rangle \quad \text{constant.}$$

$$\vec{r}\,'(t) = \langle 0,0,-gt+v \rangle$$

$$\vec{r}(t) = \langle 0,0,-\tfrac{1}{2}gt^2+vt \rangle$$

$$KE(t) = \frac{1}{2}m\|\vec{r}\,'(t)\|^2$$

$$= \frac{1}{2} m \left[0^2 + 0^2 + (-gt + v)^2 \right]$$

$$= \frac{1}{2} m \left[g^2 t^2 - 2gvt + v^2 \right]$$

$$= \frac{1}{2} m g^2 t^2 - mgvt + \frac{1}{2} m v^2 .$$

Next: Observe that $\vec{F}$ is conservative.

$$f(x, y, z) = mgz$$

$$-\nabla f = \langle 0, 0, -mg \rangle = \vec{F} .$$

Define

$$PE(t) = f(\vec{r}(t)) .$$

$$= f(0, 0, -\tfrac{1}{2} g t^2 + vt)$$

$$= mg\left(-\tfrac{1}{2} g t^2 + vt\right)$$

$$= -\frac{1}{2} m g^2 t^2 + mgvt$$

Finally we have

$$KE(t) + PE(t) = \underbrace{\frac{1}{2}mv^2}_{\text{independent of } t.}$$

$$PE(\text{start}) = f(0,0,0) = 0$$

$$KE(\text{start}) = \frac{1}{2}m\|\vec{r}\,'(0)\|^2 = \frac{1}{2}mv^2$$

When the projectile reaches max height we get $\|\vec{r}\,'(t)\| = 0$, so $KE(\text{top}) = 0$.

$$PE(\text{top}) = \frac{1}{2}mv^2 - KE(\text{top})$$

$$PE(\text{top}) = \frac{1}{2}mv^2$$

$$\cancel{m}g\,z(\text{top}) = \frac{1}{2}\cancel{m}v^2$$

$$z(\text{top}) = \frac{1}{2g}v^2$$

This is the max height of the particle. Note: It is independent of mass!

UNITS :

$g \sim$ accel $\sim m/s^2$

$v \sim$ velocity $\sim m/s$

$$\frac{1}{2g} \cdot v^2 \sim \frac{1}{m/s^2} \cdot \left(\frac{m}{s}\right)^2 \sim m$$

So $\frac{1}{2g} v^2 \sim$ Length ✓

Back to Math.

Since $\vec{G} = \langle y, 2x+1 \rangle$ does not satisfy "independence of path", it cannot be a gradient vector field.

Is there an easier way to see this ?

Theorem (Conservative Vector Fields).

Given vector field in $\mathbb{R}^2$:

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$$

The following statements are equivalent.

- $\vec{F} = \nabla f$ for some $f(x,y)$
- $\int_{\text{loop}} \vec{F} = 0$ for any loop
- "Cross-Partial Property"
  $$P_y = Q_x$$

Check: $\vec{F}(x,y) = \langle y, x+1 \rangle$

$$P(x,y) = y$$
$$Q(x,y) = x+1$$
$$P_y = 1$$
$$Q_x = 1$$
SAME

So $\vec{F}$ is conservative.

But $\vec{G}(x,y) = \langle y, 2x+1 \rangle$

$$P_y = 1$$
$$Q_x = 2$$
NOT SAME

So $\vec{G}$ is <u>not</u> conservative.

3D Version: Given

$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

The following are equivalent:

- $\vec{F} = \nabla f$ for some $f(x,y,z)$
- $\int_{Loop} \vec{F} = 0$ for any Loop.
- $\begin{cases} P_y = Q_x \\ P_z = R_x \\ Q_z = R_y \end{cases}$ "cross-partial property"

[In Higher Dimensions:

$\vec{F}(x_1, \ldots, x_n) = \langle F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n) \rangle$

Cross-Partial property says

$$\frac{dF_i}{dx_j} = \frac{dF_j}{dx_i} \quad \text{for all } i \neq j.$$

EASY TO CHECK :)]

Example: $\quad P \quad Q \quad R$

$$\vec{F}(x,y,z) = \langle 3x^2 z,\ z^2,\ x^3 + 2yz \rangle$$

Check cross partials:

$P_y = 0$ & $Q_x = 0$ ✓

$P_z = 3x^2$ & $R_x = 3x^2$ ✓

$Q_z = 2z$ & $R_y = 2z$ ✓

This guarantees that $\vec{F}$ has an antiderivative scalar field.

How can we find it?

TWO METHODS:

(1) Try really hard.

Looking for $f(x,y,z)$ such that

$$f_x(x,y,z) = 3x^2 z$$

$$f_y(x,y,z) = z^2$$

$$f_z(x,y,z) = x^3 + 2yz$$

START:

$$f_y = z^2$$

$$f = z^2 y + g(x,z)$$

$$f_x = 3x^2 z$$

$$f_x = 0 + g_x$$

$$g_x = 3x^2 z$$

$$g = x^3 z + h(y,z)$$

Seems like we're going around in circles!

(2) Use the Fund Thm:

If $\vec{F} = \nabla f$ then

$$\int_{\text{curve}} \vec{F} = f(\text{end}) - f(\text{start}).$$

(Independent of the shape of curve.).

TRICK: Fix some start point

start = $(0,0,0)$

Consider <u>any</u> path from $(0,0,0)$ to some point (a,b,c).

Say $\vec{r}(t) = (at, bt, ct)$
$t = 0$ to 1.

Then

$$\int_{\text{curve}} \vec{F} = \underbrace{f(a,b,c)}_{\substack{\text{this is} \\ \text{what we want} \\ \text{to know.}}} - \underbrace{f(0,0,0)}_{\text{const.}}$$

So let's compute:

$$\int_0^1 \vec{F}(at, bt, ct) \cdot \langle a, b, c \rangle \, dt$$

$$= \int_0^1 \left\langle 3(at)^2(ct), (ct)^2, (at)^3 + 2(bt)(ct) \right\rangle \cdot \langle a, b, c \rangle \, dt.$$

$$= \int 3a^3ct^3 + bc^2t^2 + ca^3t^3 + 2bc^2t^2 \, dt$$

$$= 3a^3c\frac{t^4}{4} + bc^2\frac{t^3}{3} + ca^3\frac{t^4}{4} + 2bc^2\frac{t^3}{3} \Big|_0^1$$

$$= \underbrace{\frac{3}{4}a^3c + \frac{bc^2}{3} + \frac{ca^3}{4} + \frac{2bc^2}{3}}_{\text{This is our desired } f(a,b,c).}$$

In other words:

$$f(x,y,z) = \frac{3}{4}x^3z + \frac{1}{3}yz^2 + \frac{1}{4}x^3z + \frac{2}{3}yz^2.$$

$$= x^3z + yz^2$$

CHECK:

$$f(x,y,z) = x^3z + yz^2$$

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$= \langle 3x^2z, z^2, x^3 + 2yz \rangle$$

$$= \vec{F} \quad \checkmark$$

It worked.

HW 5 will be posted tomorrow, due Tuesday.

Review:

- To integrate a scalar field f along an oriented curve C:

$$\int_C f \, ds = \int f(\vec{r}(t)) \, \|\vec{r}\,'(t)\| \, dt$$

Example: Let $f(x,y) = x$ be the height of a wall above point (x,y). If the base of the wall is the curve $y = x^2$ $(0 \le x \le 1)$, find the area of the wall.

To compute this we must parametrize the curve:

$$\vec{r}(t) = \langle t, t^2 \rangle \text{ from } t=0 \text{ to } t=1.$$

Then:

$$\text{area of wall} = \int \text{area of skinny rectangle}$$

$$= \int \underset{\text{height}}{f} \, \underset{\text{length of base}}{ds}$$

$$= \int_0^1 f(\vec{r}(t)) \, \|\vec{r}\,'(t)\| \, dt$$

$$= \int_0^1 t \sqrt{1^2 + (2t)^2} \, dt$$

$$= \int_0^1 t \sqrt{1 + 4t^2} \, dt$$

$$\left[u = 1 + 4t^2, \quad du = 8t \, dt \right]$$

$$= \frac{1}{8} \int_1^5 \sqrt{u} \, du$$

$$= \frac{1}{8} \left(\frac{2}{3} u^{3/2} \right)\Big|_1^5$$

$$= \frac{1}{8}\left(\frac{2}{3}\cdot 5^{3/2} - \frac{2}{3}\right)$$

$$\approx 0.85$$

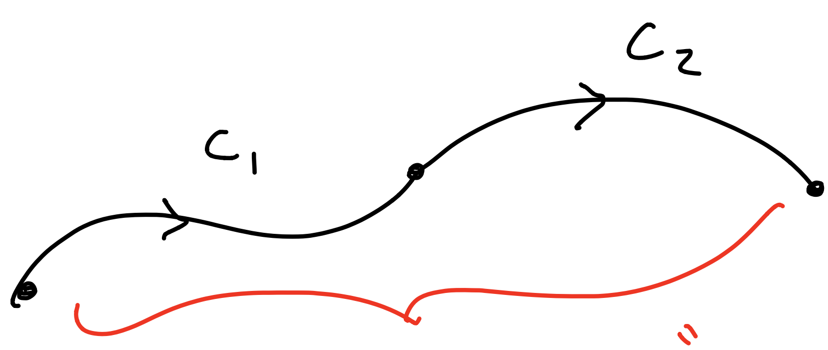
- To integrate a vector field $\vec{F}$ along an oriented curve C:

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

Example : Your increase in Kinetic energy (KE) due to the force $\vec{F}$.

- Reversing & Concatenating Curves :

Curves can be "added & subtracted"



total curve is called "$C_1 + C_2$"

The reverse orientation of a curve C is called "$-C$".

Theorem:

$$\text{"}\int_{C_1+C_2}\text{"} = \text{"}\int_{C_1}\text{"} + \text{"}\int_{C_2}\text{"}$$

$$\text{"}\int_{-C}\text{"} = \text{"}-\int_{C}\text{"}$$

[Compare: $\text{"}\int_a^b\text{"} = \text{"}-\int_b^a\text{"}$.]

- Fundamental Theorem of Line Integrals:

$$\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = f(\vec{r}(b)) - f(\vec{r}(a))$$

[does not depend on the "shape" of the curve; only the endpoints.]

In particular, if you follow a loop ($\vec{r}(a) = \vec{r}(b)$) then

$$\int_{\text{loop}} \nabla f \cdot \vec{T}\, ds = 0$$

- Conservation of Energy:

If $\vec{F} = -\nabla f$ then we can think of $\vec{F}$ as a "force field" and f as the "potential energy" (PE).

[The force is trying to decrease your potential energy!]

Then

$$\int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = -\left[f(\vec{r}(b)) - f(\vec{r}(a)) \right]$$

increase in KE = decrease in PE

In other words, the "total

mechanical engergy" KE + PE

is conserved over time.

For this reason, a vector field

of the form $\vec{F} = \nabla f$ (or $-\nabla f$)

is called "conservative".

- Not all vector fields are conservative.

Example: Let $\vec{F}(x,y) = \langle -y, x \rangle$.

We will show that $\vec{F}$ is not cons.

by finding a loop C such that

$$\int_C \vec{F} \cdot \vec{T} \, ds \neq 0$$

Here's the loop:

$\vec{r}(t) = \langle \cos t, \sin t \rangle$, $t = 0$ to 2π

Then

$$\int \vec{F} \cdot \vec{T} \, ds = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt$$

$$= \int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t) \, dt$$

$$= \int_0^{2\pi} 1 \, dt = 2\pi \neq 0 .$$

Conclusion: There is <u>no</u> function $f(x,y)$ such that

$$\vec{F} = \nabla f$$

$$\langle -y, x \rangle = \langle f_x, f_y \rangle .$$

Conversely, if $\vec{F}$ is any vector field satisfying

$$\int_{\text{Loop}} \vec{F} \cdot \vec{T} \, ds = 0$$

for *every* loop, then I claim that $\vec{F}$ is conservative, i.e., there exists a scalar field f such that

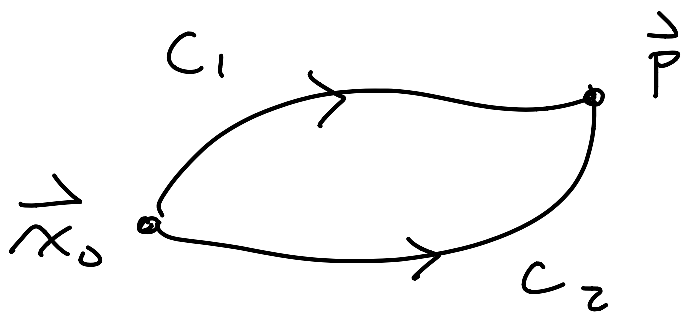
$$\vec{F} = \nabla f.$$

Idea: Fix an arbitrary basepoint $\vec{x}_0$. Then for any point $\vec{p}$ we define

$$f(\vec{p}) = \int_{\vec{x}_0}^{\vec{p}} \vec{F} \cdot \vec{T}\, ds,$$

where the integral is taken along <u>any path</u> from $\vec{x}_0$ to $\vec{p}$.

To see that this makes sense, consider two different paths



Then "$C_1 - C_2$" is a <u>loop</u> so that

$$\int_{C_1 - C_2} \vec{F} \cdot \vec{T}\, ds = 0$$

$$\int_{C_1} \vec{F} \cdot \vec{T}\, ds - \int_{C_2} \vec{F} \cdot \vec{T}\, ds = 0$$

$$\int_{C_1} \vec{F} \cdot \vec{T}\, ds = \int_{C_2} \vec{F} \cdot \vec{T}\, ds \quad \checkmark$$

[Recall from Calc I:

$$\frac{d}{dx} \int_{x_0}^{x} f(t)\, dt = f(x). \text{]}$$

Unfortunately, it is very hard to check that $\int \vec{F} \cdot \vec{T}\, ds = 0$ around <u>any</u> loop. [There are infinitely many possible loops!]

It turns out there is an easier way to determine if a vector field is conservative.

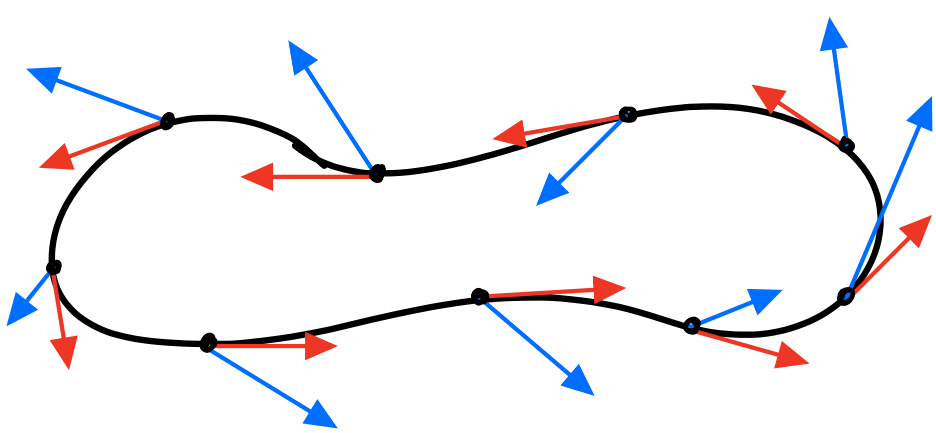
Idea: If $\vec{F} = \nabla f$ and f represents "height" then $\vec{F}$ points "uphill".

If $\int_{\text{loop}} \vec{F} \cdot \vec{T}\, ds \neq 0$ (say > 0)

then this is a contradiction because you walked around a loop but you ended up "higher" than you started.

For a general field $\vec{F}$ we get

$\int \vec{F} \cdot \vec{T}\, ds > 0$ when $\vec{F}$ (on average) points in the direction $\vec{T}$ of the loop:



This "curling" of a vector field F around a loop can be precisely measured by the "curl" $\nabla \times \vec{F}$.

[Warning: This concept is only defined in $\mathbb{R}^3$, and partially in $\mathbb{R}^2$.]

Given a vector field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle,$$

we define the "curl"

$$\nabla \times \vec{F} = \text{"} \langle \partial_x, \partial_y, \partial_z \rangle \times \langle P, Q, R \rangle \text{"}$$

$$= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

Theorem (Cross - Partial Property of Conservative Vector Fields):

- Given a vector field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$,

$\vec{F}$ is conservative $\iff$ $\nabla \times \vec{F} = \langle 0,0,0 \rangle$

$$\iff \begin{cases} R_y = Q_z \\ P_z = R_x \\ Q_x = P_y \end{cases}$$

- Given a vector field $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ written as $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$,

$\vec{F}$ is conservative $\iff$ $Q_x = P_y$

Remarks: These criteria are very easy to check! :)

Example: Prove that the vector field

$$\vec{F}(x,y,z) = \langle \underset{P}{3x^2 z}, \underset{Q}{z^2}, \underset{R}{x^3 + 2yz} \rangle$$

is conservative.

Proof : Check the "cross-partials"

$$R_y = 0 + 2z \, , \quad Q_z = 2z \qquad \checkmark$$

$$P_z = 3x^2 \, , \quad R_x = 3x^2 + 0 \qquad \checkmark$$

$$Q_x = 0 \, , \quad P_y = 0 \qquad \checkmark$$

Done. ///

This proves that an "antiderivative" $f(x,y,z)$ exists such that

$$\vec{F} = \nabla f$$

but it does not tell us how to find f.

In this case I can tell you that

$$f(x,y,z) = x^3 z + y z^2 + \text{constant}$$

because I planned it that way!

In general it is hard to find antiderivatives.

HW5 is up; due Tues.

Review:

- A vector field $\vec{F}$ is called "conservative" if it is a gradient,

$$\vec{F} = \nabla f.$$

That is, if $\vec{F}$ has an "antiderivative".

[Example: If $\vec{F} = -\nabla f$ is a conservative force field then the scalar field f is called "potential energy".]

- Given vector field $\vec{F} = \langle P, Q, R \rangle$ in 3D space we define the "curl"

$$\nabla \times \vec{F} = \text{``} \langle \partial_x, \partial_y, \partial_z \rangle \times \langle P, Q, R \rangle \text{''}$$

$$= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle,$$

which is also a vector field in 3D.

Given any 2D vector field $\vec{F} = \langle P, Q \rangle$ we can also define the "curl" of $\vec{F}$,

$$\operatorname{curl}(\vec{F}) = Q_x - P_y,$$

which is a <u>scalar</u> field!

Remarks:

- We can think of $\vec{F} = \langle P, Q \rangle$ as the shadow of $\vec{F} = \langle P, Q, 0 \rangle$, so

$$\nabla \times \vec{F} = \langle 0, 0, Q_x - P_y \rangle.$$

Idea: The curl of a 2D vector field $\langle P, Q \rangle$ points "up" into the z-direction.

- There is no concept of "curl" outside of 2D & 3D space. ///

- "Cross-Partial" Property of Conservative Vector Fields.

In $\mathbb{R}^3$:

vector

$\vec{F}$ is cons. $\iff$ $\nabla \times \vec{F} = \langle 0, 0, 0 \rangle$

$$\iff \begin{cases} R_y = Q_z \\ P_z = R_x \\ Q_x = P_y \end{cases}$$

In $\mathbb{R}^2$:

scalar

$\vec{F}$ is cons. $\iff$ $\text{curl}(\vec{F}) = 0$

$\iff$ $Q_x = P_y$

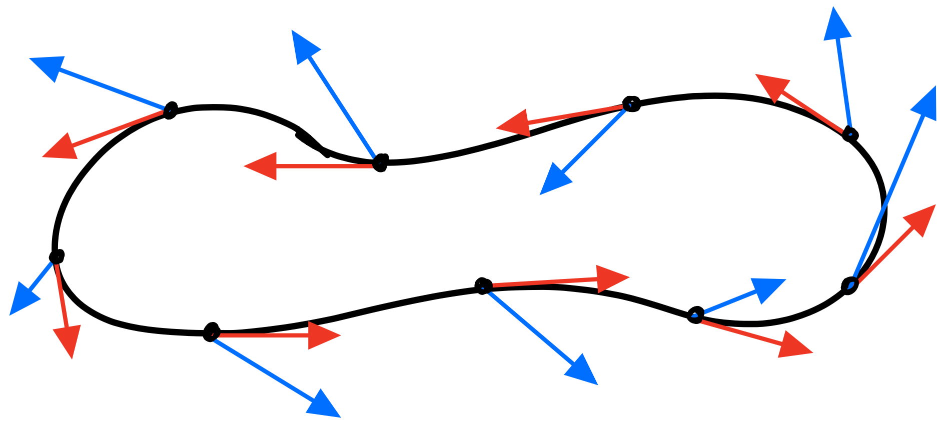
Intuition: If $\vec{F} = \nabla f$ then

$$\oint_{\text{loop}} \vec{F} \cdot \vec{T}\, ds = 0 \quad \text{for } \underline{\text{any}} \text{ loop.}$$

If $\text{curl}(\vec{F}) \neq 0$ then we can find some loop C such that

$$\oint_C \vec{F} \cdot \vec{T}\, ds > 0$$

Picture:



This vector field "curls" with the loop.

Example: $\vec{F}(x,y) = \langle -y, x \rangle$.

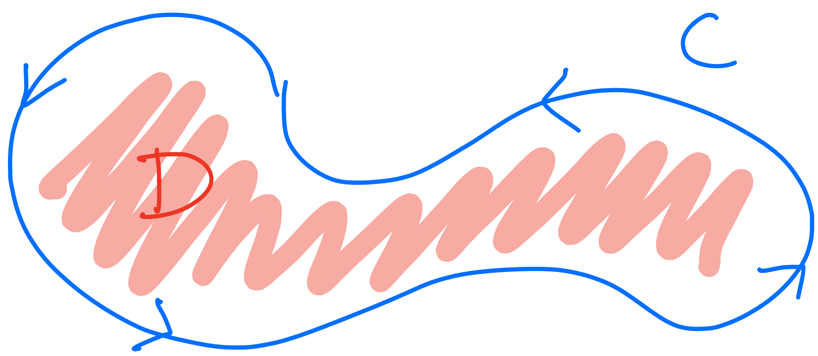
We saw that this vector field "curls" counterclockwise. And, indeed, we have

$$\begin{aligned} \mathrm{curl}(\vec{F}) &= Q_x - P_y \\ &= \partial_x(x) - \partial_y(-y) \\ &= 1 + 1 = 2 > 0 \end{aligned}$$

The next theorem will make this intuition precise.

Green's Theorem:

Let C be a "simple closed curve" (no self-intersections), oriented counterclockwise. Let D be the interior 2D region:



[Mnemonic: Region D is always "to the left" of the curve C.]

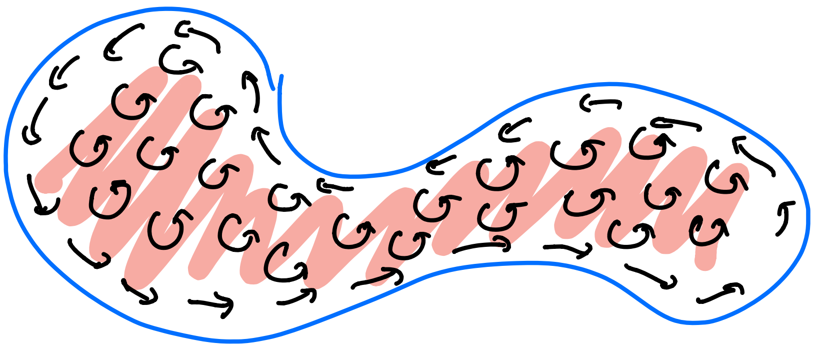
If $\vec{F} = \langle P, Q \rangle$ is a 2D field defined at every point of D, then

$$\underbrace{\iint_D \mathrm{curl}(\vec{F})\,dA}_{\text{amount that } \vec{F} \text{ is ``curling'' inside the loop}} = \underbrace{\oint_C \vec{F}\cdot\vec{T}\,ds}_{\text{amount that } \vec{F} \text{ points ``along'' the loop}}$$

This is the precise meaning of curl:

"$\mathrm{curl}(\vec{F})\,dA$" = how much is $\vec{F}$ pointing along a tiny c.c.w. loop?

The proof is hard, but here is the idea:



All the internal rotations cancel each-other and all that's left is the circulation along the boundary. ///

Some other notations:

$$\iint \operatorname{curl}(\vec{F})\,dA = \oint \vec{F}\cdot d\vec{r}$$

$$\iint (Q_x - P_y)\,dx\,dy = \oint \langle P, Q\rangle \cdot \left\langle \frac{dx}{\cancel{dt}}, \frac{dy}{\cancel{dt}} \right\rangle \cancel{dt}$$

$$\boxed{\iint (Q_x - P_y)\,dx\,dy = \oint P\,dx + Q\,dy}$$

This is just a notation. It means exactly the same thing.

One more observation:

If $\operatorname{curl}(\vec{F}) = 0$ everywhere, then for any loop C we have

$$\oint \vec{F}\cdot\vec{T}\,ds = \iint 0\,dA = 0,$$

so that $\vec{F}$ is conservative.

Thus the "cross-partial" property is proved ✓

As with any "Fundamental Theorem

of Calculus", Green's Theorem can be used to simplify computations.

Example: Let C be the c.c.w. oriented perimeter of the rectangle

$$0 \leq x \leq 2,$$
$$0 \leq y \leq 3.$$

Compute the circulation of the vector field $\vec{F} = \langle P, Q \rangle = \langle x^2 y, y-3 \rangle$ around C.

To do this directly you need to parametrize all four sides of the rectangle and compute four line integrals. Instead, we will use Green's Theorem and integrate the curl $Q_x - P_y = 0 - x^2$ not conservative over the <u>interior</u> of the rectangle:

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \iint_D \operatorname{curl}(\vec{F})\, dA$$

$$= \iint_D -x^2\, dx\, dy$$

$$= \int_0^3 \int_0^2 -x^2\, dx\, dy$$

$$= \int_0^3 -\frac{8}{3}\, dy$$

$$= 3\left(-\frac{8}{3}\right) = -8.$$

The force drained 8 units of KE while you traveled ccw around the perimeter. Or, your airplane needed 8 gallons of fuel to oppose the wind $f(x,y) = \langle x^2y, y-3 \rangle$.

But be careful!

WARNING: Consider the field

$$\vec{F}(x,y) = \frac{1}{x^2+y^2}\langle -y, x\rangle$$

$$= \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

We have

$$Q_x = \partial_x \left(\frac{x}{x^2+y^2} \right)$$

$$= \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$$

$$= (y^2-x^2)/(x^2+y^2)^2$$

and

$$P_y = \partial_y \left(\frac{-y}{x^2+y^2} \right)$$

$$= \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2}$$

$$= (y^2 - x^2) \big/ (x^2+y^2)^2$$

$$= Q_x \quad (\text{surprise!})$$

So that $\text{curl}(\vec{F}) = Q_x - P_y = 0$.

However, I claim that

$$\oint_{\substack{\text{unit}\\\text{circle}}} \vec{F} \cdot \vec{T}\, ds \neq 0.$$

To ~~see~~ this, we can use the standard parametrization:

$$\vec{r}(t) = \langle \cos t, \sin t \rangle, \quad t = 0 \text{ to } 2\pi.$$

$$\vec{F}(\vec{r}(t)) = \frac{1}{\cos^2 t + \sin^2 t} \langle -\sin t, \cos t \rangle$$

$$= \langle -\sin t, \cos t \rangle$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$$

and hence

$$\oint_{\vec{r}} \vec{F} \cdot d\vec{r} = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt$$

$$= \int_0^{2\pi} 1\, dt$$

$$= 2\pi \neq 0 .$$

What went wrong ?

Green's Theorem assumes that curl ($\vec{F}$) <u>exists</u> at every point inside the loop. But this field $\vec{F}(x,y) = \langle -y, x \rangle / (x^2+y^2)$, and hence its curl, is <u>not</u> defined at the origin. So Green's Theorem doesn't apply to this calculation.

Nevertheless, Green's Theorem does give us a simplification.
For any (simple, closed, ccw) loop C
I claim that

$$\oint_C \vec{F} \cdot \vec{T} \, ds = \begin{cases} 0 & \text{if } C \text{ does not contain } (0,0) \\ 2\pi & \text{if } C \text{ contains } (0,0) \end{cases}$$

So this integral doesn't depend on the *shape* of the loop, but it *does* detect whether the loop goes around the origin. Strange!

I'll explain this next time ...

HW 5 due tomorrow.

Quiz 5 Wed.

Final Proj due Friday.

Last time we discussed the most basic form of Green's Theorem.

Today I'll present a more general form.

Green's Theorem :

Let D be a 2D region with "boundary curve" ∂D. If a vector field $\vec{F} = \langle P, Q \rangle$ is defined at every point of D, then

$$\iint_D \operatorname{curl}(\vec{F})\, dA = \oint_{\partial D} \vec{F} \cdot \vec{T}\, ds$$

$$\iint_D (Q_x - P_y)\,dx\,dy$$

$$= \oint_{\partial D} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

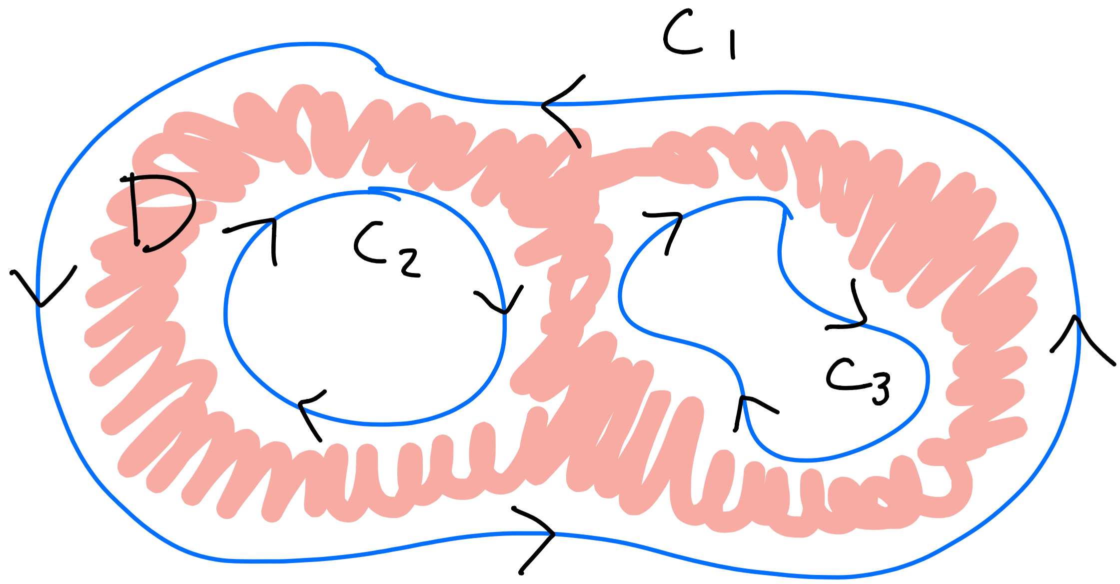
$$= \oint_{\partial D} \langle P, Q \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle dt$$

$$= \oint_{\partial D} P\,dx + Q\,dy$$

These are just different notations for the same idea:

integral of curl($\vec{F}$) over D = circulation of $\vec{F}$ along ∂D

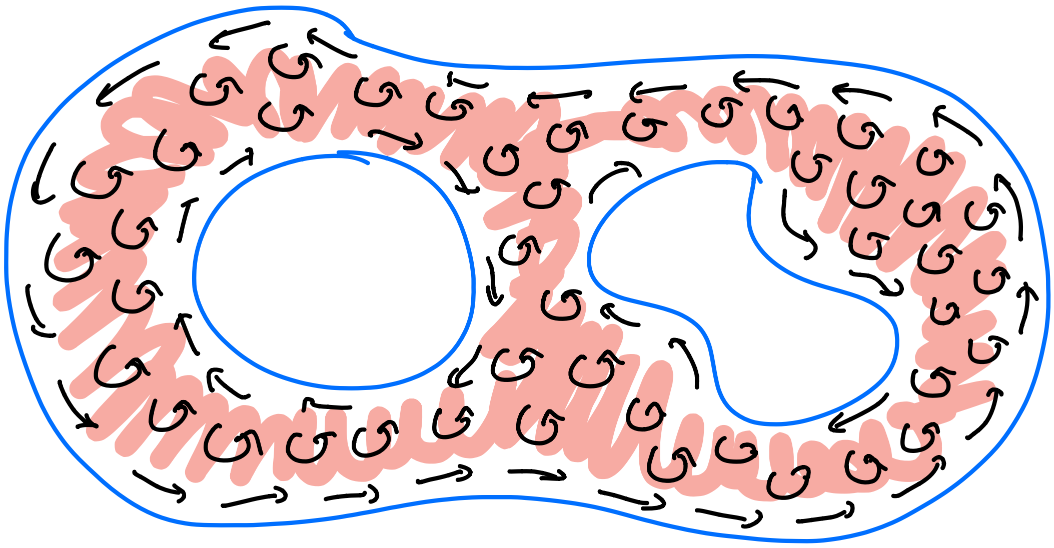
What makes this form more general is that the "boundary curve" ∂D is allowed to have multiple pieces:



In this picture, the boundary is the "sum" of three curves:

$$\partial D = C_1 + C_2 + C_3$$

The only rule is that the curves are oriented so that the region D is always "to the left". Then the idea of the proof is the same as before:



The rotations in the interior cancel, leaving only the circulation along the boundary.

Example: Consider the vector field

$$\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

We showed last time that

$$\operatorname{curl}(\vec{F})(x,y) = \begin{cases} 0 \text{ if } (x,y) \neq (0,0) \\ \text{undefined if } (x,y) = (0,0) \end{cases}$$

If C is a simple, connected, counterclockwise loop then I claim

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \begin{cases} 2\pi & \text{if } C \text{ contains } (0,0) \\ 0 & \text{if } C \text{ does not contain } (0,0) \end{cases}$$

Proof: If C does not contain $(0,0)$ then $\operatorname{curl}(\vec{F}) = 0$ at every point inside the loop, so Green's Theorem says

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \iint_{\text{inside}} 0\, dA = 0.$$

For the other statement, let C_1 & C_2 be any two loops containing $(0,0)$.

Here is a picture, assuming that the two loops do not intersect:

If D is the region between the curves then we must have

$$\partial D = C_1 - C_2$$

we need to reverse the orientation of C_2 so that D is always "to the left" of ∂D

Then since $\text{curl}(\vec{F}) = 0$ at every point of D, Green's Theorem says

$$0 = \iint_D \text{curl}(\vec{F})\, dA$$

$$= \int_{C_1 - C_2} \vec{F} \cdot \vec{T} \, ds$$

$$= \int_{C_1} \vec{F} \cdot \vec{T} \, ds - \int_{C_2} \vec{F} \cdot \vec{T} \, ds$$

and hence

$$\int_{C_1} \vec{F} \cdot \vec{T} \, ds = \int_{C_2} \vec{F} \cdot \vec{T} \, ds$$

[If the curves C_1 & C_2 intersect then this is still true but the picture is more complicated.]

Thus we only need to compute the circulation around one specific curve that contains $(0,0)$. The easiest choice is the unit circle C :

$$\vec{r}(t) = \langle \cos t, \sin t \rangle,$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle,$$

$$\vec{F}(\vec{r}(t)) = \frac{1}{\cos^2 t + \sin^2 t} \langle -\sin t, \cos t \rangle$$

$$= \langle -\sin t, \cos t \rangle,$$

so that

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt$$

$$= \int_0^{2\pi} 1\, dt = 2\pi \quad \checkmark$$

That's pretty amazing! Today we will see what this result has to do with gravity & electromagnetism.

The "Flux Form" of Green's Theorem:

Given a vector field $\vec{F} = \langle P, Q \rangle$, we may consider the vector field

$$\vec{G} = \langle U, V \rangle = \langle -Q, P \rangle$$

[We rotated $\vec{F}$ by $90°$.] Let's apply Green's Theorem to $\vec{G}$:

$$\iint_D (V_x - U_y)\,dx\,dy = \oint_{\partial D} U\,dx + V\,dy$$

$$\iint_D (P_x + Q_y)\,dx\,dy = \oint_{\partial D} -Q\,dx + P\,dy$$

$$\iint_D \nabla \cdot \langle P, Q \rangle\,dx\,dy = \int_{\partial D} \langle P, Q \rangle \cdot \langle dy, -dx \rangle$$

"$\iint_D$ divergence = Flux across ∂D"

WHAT ?

Given a vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$
we define a scalar field $\nabla \cdot \vec{F} : \mathbb{R}^n \to \mathbb{R}$
called the "divergence of $\vec{F}$":

$$\nabla \cdot \vec{F} = \text{“}\langle \partial_{x_1}, \dots, \partial_{x_n} \rangle \cdot \langle P_1, \dots, P_n \rangle\text{”}$$

$$= \underbrace{\frac{dP_1}{dx_1} + \frac{dP_2}{dx_2} + \cdots + \frac{dP_n}{dx_n}}_{\text{this is a scalar field}}$$

Sometimes we also write

$$\nabla \cdot \vec{F} = \text{“div}(\vec{F})\text{”}$$

Special Cases:

- $$\vec{F} = \langle P, Q, R \rangle$$
  $$\nabla \cdot \vec{F} = P_x + Q_y + R_z$$

- $$\vec{F} = \langle P, Q \rangle$$
  $$\nabla \cdot \vec{F} = P_x + Q_y$$

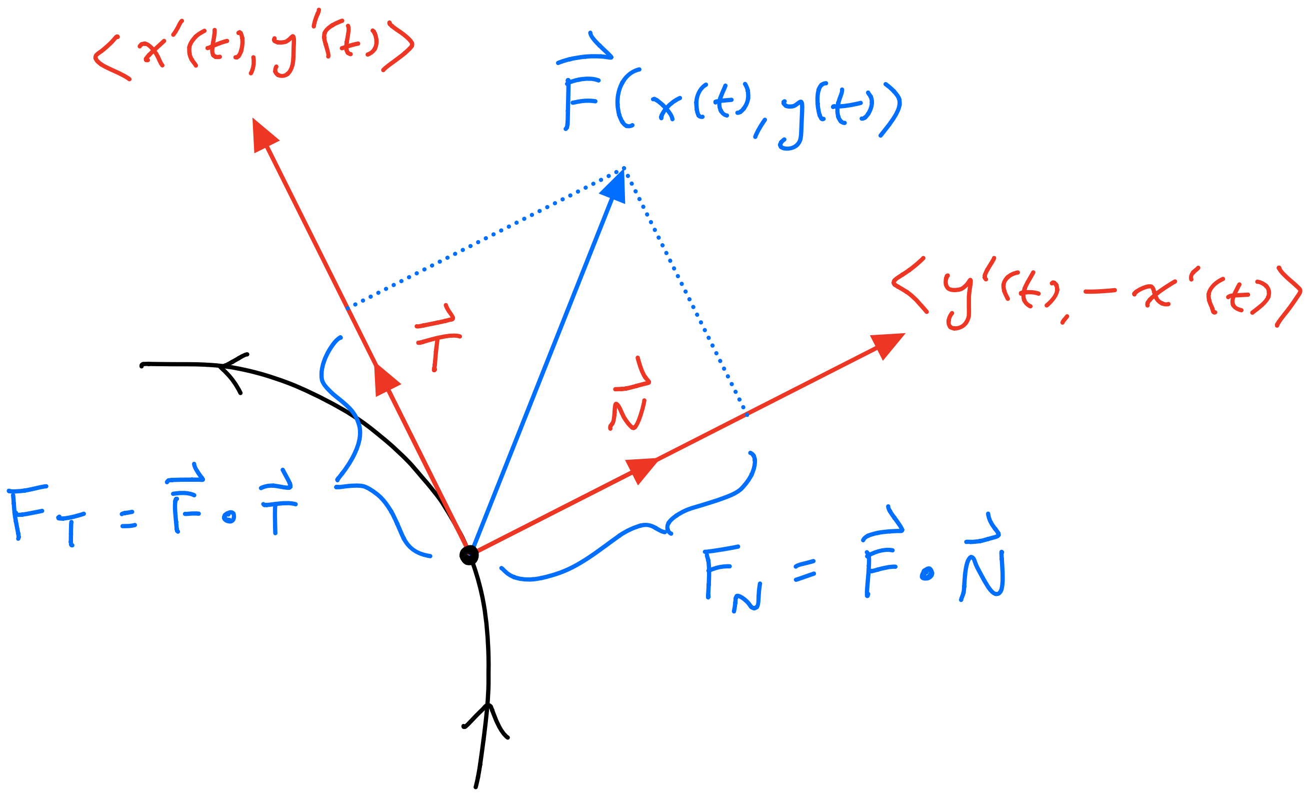
The flux form of Green's Theorem involves the divergence:

$$\iint_D \operatorname{div}(\vec{F})\, dA = \int_{\partial D} \langle P, Q \rangle \cdot \langle dy, -dx \rangle$$

$$= \underbrace{\int_{\partial D} \vec{F}(\vec{r}(t)) \cdot \left\langle \frac{dy}{dt}, -\frac{dx}{dt} \right\rangle dt}$$

This is called the "Flux of $\vec{F}$ across the curve ∂D"

Picture:



Given a parametrized curve $\vec{r}(t) = \langle x(t), y(t)\rangle$ we have a velocity $\vec{r}'(t) = \langle x'(t), y'(t)\rangle$ & a unit tangent vector

$$\vec{T}(\vec{r}(t)) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}.$$

To compute the circulation of a

vector field $\vec{F}$ along $\vec{r}(t)$ we observe that the component of $\vec{F}$ in the tangent direction is :

$$F_T = \vec{F} \cdot \vec{T}$$

[See HW 5.1]

The integral of $\vec{F}$ along $\vec{r}(t)$ is

$$\int_C \vec{F} \cdot \vec{T} \, ds = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt$$

We also obtain a unit vector in the normal direction by rotating $\vec{T}$ $90°$ clockwise. In terms of the parametrization :

$$\vec{N}(\vec{r}(t)) = \frac{\langle \vec{y}\,'(t), -x'(t) \rangle}{\| \langle \vec{y}\,'(t), -x'(t) \rangle \|} .$$

And the component of $\vec{F}$ in the normal direction is

$$F_N = \vec{F} \cdot \vec{N}$$

We define the "flux of $\vec{F}$ across the curve $\vec{r}(t)$" as the integral of the normal component:

$$\int_C \vec{F} \cdot \vec{N} \, ds = \int \vec{F}(\vec{r}(t)) \cdot \langle y'(t), -x'(t) \rangle \, dt$$

Meaning: How much is $\vec{F}$ pointing perpendicular to (specifically, to the right of) the curve?

To understand the flux form of Green's Theorem, suppose that $\vec{F}$ is the velocity field of a fluid (liquid or gas). Then

$$\iint_D \nabla \cdot \vec{F}\, dA = \int_{\partial D} \vec{F} \cdot \vec{N}\, ds$$

how much does the fluid expand in the region D? = how much fluid flows across the boundary ∂D?

That makes sense!

Examples of Divergence:

- Fluid Dynamics: If $\vec{F}$ is the velocity field of a fluid then

$\nabla \cdot \vec{F}$ = infinitesimal amount of expansion/contraction at a point

We often assume that the flow is "incompressible":

$$\nabla \cdot \vec{F} = 0.$$

[This is closely related to "conservation of mass": no fluid is created or destroyed.]

- Gauss' Law for electric & gravitational forces.

  - Let $\rho(x,y,z)$ be a distribution of charge and let $\vec{E} : \mathbb{R}^3 \to \mathbb{R}^3$ be the force exerted on a unit point charge by the charges ρ. Then

$$\nabla \cdot \vec{E} = \rho$$

  - Let $\rho(x,y,z)$ be a distribution of mass and let $\vec{g} : \mathbb{R}^3 \to \mathbb{R}^3$ be the force exerted on a unit point mass by the masses ρ. Then

$$\nabla \cdot \vec{g} = -\rho$$

See HW 5.5 for a 2D example.

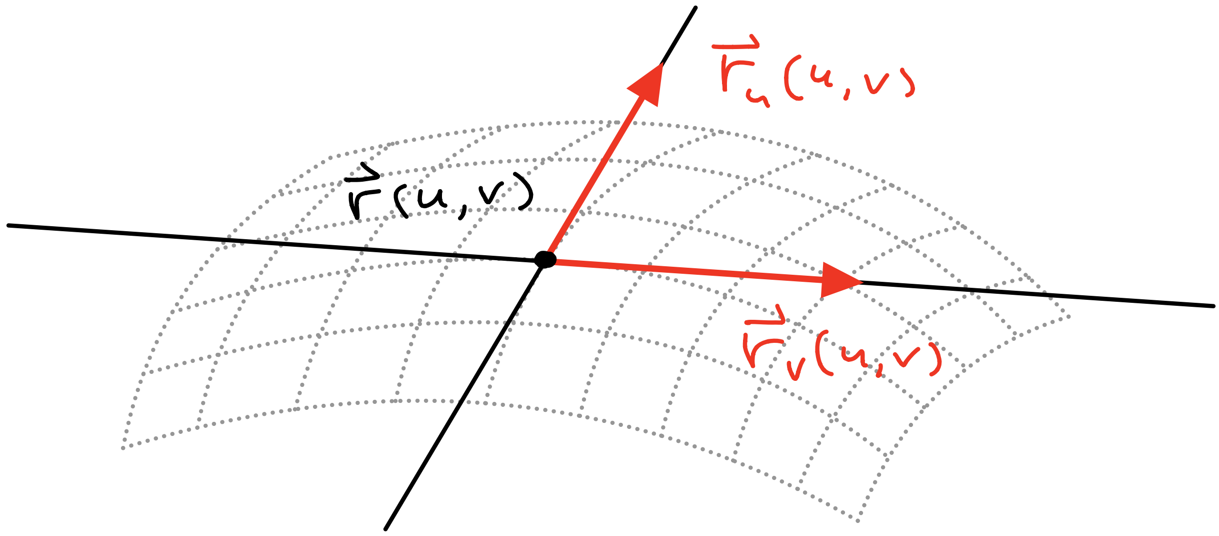
To apply these ideas in 3D we need to discuss "flux across a 2D surface in $\mathbb{R}^3$".

Goal: Integrate a scalar or vector field over a 2D surface in $\mathbb{R}^3$.

How?

We must first <u>parametrize</u> the surface. We can think of a "parametrized surface in $\mathbb{R}^3$" as a function $\vec{r} : \mathbb{R}^2 \to \mathbb{R}^3$:

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle .$$



At each point $\vec{r}(u,v)$ we have two basic "velocity vectors"

$$\vec{r}_u = \langle x_u, y_u, z_u \rangle$$
$$\vec{r}_v = \langle x_v, y_v, z_v \rangle$$

The area of a tiny parallelogram near the point $\vec{r}(u,v)$ is the length of a cross product:

$$\underbrace{dS}_{\text{tiny piece of area on the surface}} = \| (\underbrace{\vec{r}_u\, du}) \times (\underbrace{\vec{r}_v\, dv}) \|$$

vectors generating a tiny parallelogram on the surface

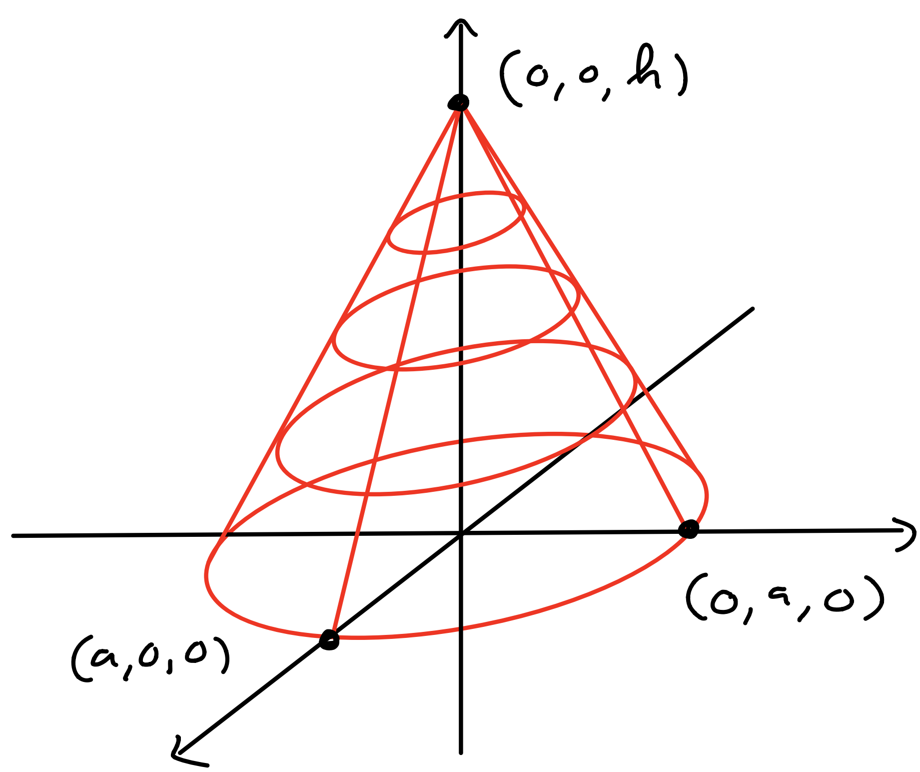
$$= \|\vec{r}_u \times \vec{r}_v\| \, du\, dv$$

This is how we compute the surface area of a parametrized 2D surface in $\mathbb{R}^3$:

$$\text{surface area} = \iint dS$$

$$= \iint \|\vec{r}_u \times \vec{r}_v\| \, du\, dv$$

Example: Surface area of a cone with height h & radius a:



To parametrize the surface it is convenient to use polar coordinates:

$$\vec{r}(r,\theta) = \left\langle r\cos\theta,\ r\sin\theta,\ \frac{h}{a}(a-r) \right\rangle$$

$$\vec{r}_r = \langle \cos\theta,\ \sin\theta,\ -h/a \rangle$$

$$\vec{r}_\theta = \langle -r\sin\theta,\ r\cos\theta,\ 0 \rangle$$

$$\vec{r}_r \times \vec{r}_\theta = \left\langle \frac{h}{a} r\cos\theta,\ \frac{h}{a} r\sin\theta,\ r \right\rangle$$

$$= r\left\langle \frac{h}{a}\cos\theta,\ \frac{h}{a}\sin\theta,\ 1 \right\rangle$$

$$\|\vec{r}_r \times \vec{r}_\theta\| = r\sqrt{\left(\frac{h}{a}\right)^2 + 1}$$

So the surface area is

$$\iint_{\text{cone}} dS = \iint \|\vec{r}_r \times \vec{r}_\theta\|\, dr\, d\theta$$

$$= \iint r\sqrt{\left(\frac{h}{a}\right)^2 + 1}\, dr\, d\theta$$

$$= \sqrt{\left(\frac{h}{a}\right)^2 + 1} \int_0^{2\pi} d\theta \int_0^a r\, dr$$

$$= 2\pi \cdot \frac{1}{2} a^2 \cdot \sqrt{\left(\frac{h}{a}\right)^2 + 1}$$

$$= \pi a \sqrt{h^2 + a^2}$$

[See page 764 of the textbook.]

Finally, let's check that our method gives the correct formula for the surface area of a sphere of radius a.

Let's use spherical coordinates:

$$\vec{r}(\theta, \varphi) = \langle a \cos\theta \sin\varphi, a \sin\theta \sin\varphi, a \cos\varphi \rangle$$

where $0 \leq \theta \leq 2\pi$,
$0 \leq \varphi \leq \pi$.

$$\vec{r}_\theta = \langle -a \sin\theta \sin\varphi, a \cos\theta \sin\varphi, 0 \rangle$$

$$\vec{r}_\varphi = \langle a \cos\theta \cos\varphi, a \sin\theta \cos\varphi, -a \sin\varphi \rangle$$

$$\vec{r}_\theta \times \vec{r}_\varphi$$

$$= \langle -a^2\cos\theta\sin^2\varphi,\ a^2\sin\theta\sin^2\varphi,\ -a^2\sin\varphi\cos\varphi \rangle$$

$$\|\vec{r}_\theta \times \vec{r}_\varphi\| = \text{computations}$$

$$= a^2\sin\varphi .$$

So the surface area is

$$\iint_{\text{sphere}} dS = \iint \|\vec{r}_\theta \times \vec{r}_\varphi\| \, d\theta \, d\varphi$$

$$= \iint a^2 \sin\varphi \, d\theta \, d\varphi$$

$$= a^2 \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \, d\varphi$$

$$= a^2 \cdot 2\pi \cdot (-(-1)+1)$$

$$= 4\pi a^2 \quad \checkmark$$

HW 5 due Tues.
(Note: New Problem!)

Recall the concept of a conservative vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ in n-dimensional space. Write

$$\vec{F}(x_1, \dots, x_n) = \langle F_1(x_1, \dots, x_n), F_2(x_1, \dots, x_n), \vdots, F_n(x_1, \dots, x_n) \rangle$$

The following statements are equivalent:

① $\vec{F} = \nabla f$ for some $f(x_1, \dots, x_n)$

(i.e., $F_i(x_1, \dots, x_n) = f_{x_i}(x_1, \dots, x_n)$)

② $\oint_C \vec{F} = 0$ for any loop C.

[$\oint$ means integral around closed loop] i.e. $\displaystyle\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = 0$

(3) Cross-Partial Property:

$$\frac{dF_i}{dx_j} = \frac{dF_j}{dx_i} \quad \text{for all } i \neq j.$$

A vector field satisfying these conditions is called "conservative".

Main Example:
Gravitational field, say force of gravity on a planet due to the sun. On HW 3 we saw

$$\vec{F}(\vec{r}(t)) = -\frac{\overbrace{\cancel{GMm}}^{\text{assume } 1}}{\|\vec{r}(t)\|^3}\,\vec{r}(t)$$

$$\vec{F}(x,y,z) = -\frac{1}{(x^2+y^2+z^2)^{3/2}}\langle x, y, z\rangle$$

$$= \langle P, Q, R \rangle.$$

So $P(x,y,z) = -x / (x^2+y^2+z^2)^{3/2}$

One can check that cross-partial property is satisfied:

$$P_y = Q_x \quad \& \quad P_z = R_x \quad \& \quad Q_z = R_y .$$

Furthermore, one can show that

$$\vec{F}(x,y,z) = -\nabla f$$

where $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

is called the gravitational potential.

Today we'll focus on 3D case:

$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$$
$$\vec{F}(x_1,x_2,x_3) = \langle F_1(x_1,x_2,x_3), F_2(x_1,x_2,x_3), F_3(x_1,x_2,x_3) \rangle$$

Consider the "nabla operator"

$$\nabla = \left\langle \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \right\rangle = \left\langle \frac{d}{dx_1}, \frac{d}{dx_2}, \frac{d}{dx_3} \right\rangle$$

[Warning: Not really a vector.]

Recall the gradient

$$\nabla f = \underbrace{\left\langle \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \right\rangle}_{\text{"vector"}} \underbrace{f}_{\text{scalar}}$$

CUTE

$$= \left\langle \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right\rangle$$

Use a similar mnemonic to define the "curl" of $\vec{F}$:

$$\nabla \times \vec{F} = \left\langle \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \right\rangle \times \langle P, Q, R \rangle$$

$$= \left\langle \frac{dR}{dy} - \frac{dQ}{dz}, \frac{dP}{dz} - \frac{dR}{dx}, \frac{dQ}{dx} - \frac{dP}{dy} \right\rangle$$

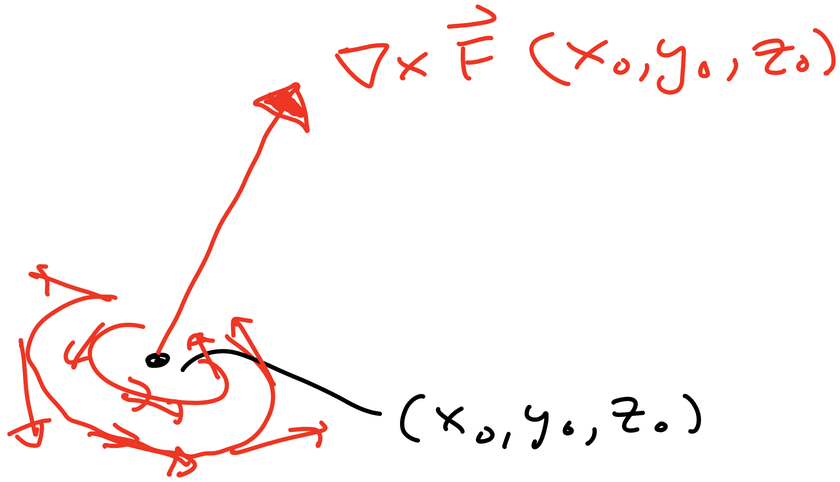
$$= \langle \underbrace{R_y - Q_z, P_z - R_x, Q_x - P_y}_{\text{this is a vector field}} \rangle$$

$\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3 \rightsquigarrow \nabla \times \vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$

vector field $\rightsquigarrow$ another vector field.

What does it mean?

It represents the amount & the direction of "rotation" in the vector field $\vec{F}$:

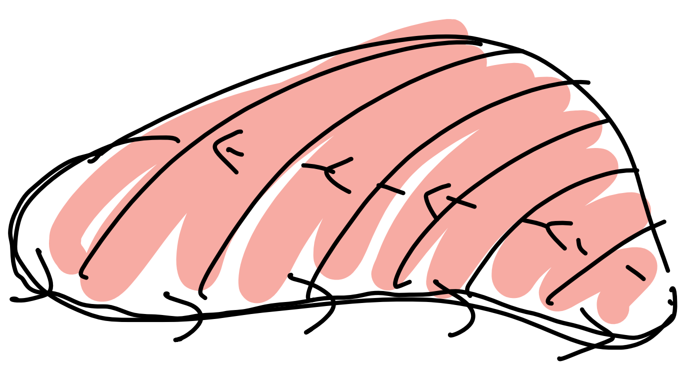


This intuition is based on a Theorem, called <u>Stokes' Theorem</u>.

$$\iint \nabla \times \vec{F} = \int \vec{F}$$

2D surface in 3D

boundary curve of the surface



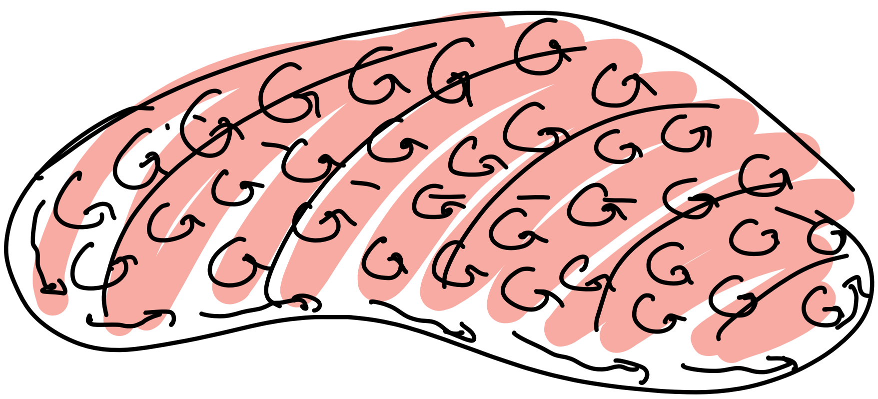
Boundary curve is oriented so surface is "to the left"

TWO QUESTIONS:

- How to define the integral of $\nabla \times \vec{F}$ over a surface?

- Why is it true?

Fake Proof:



All the little rotations cancel, except at the boundary.

How to define $\iint \nabla \times \vec{F}$?

More generally we will define

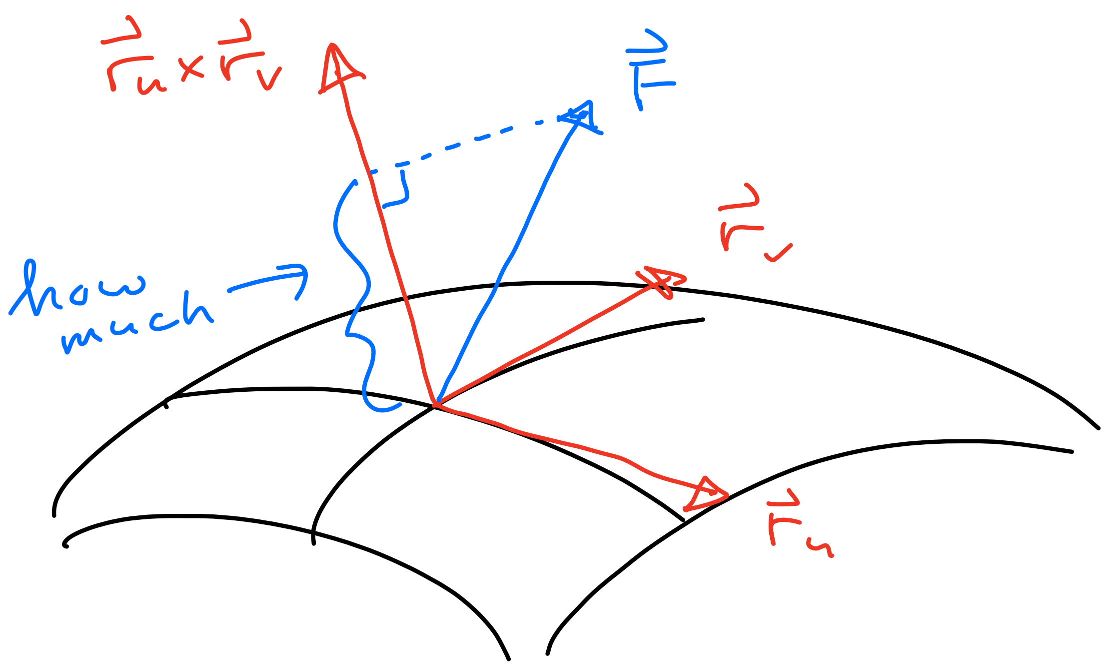
$$\iint_{\text{surface in 3D}} \vec{F}$$

vector field.

Recall: We already know how to integrate a scalar over a surface

$$\iint f(\vec{r}(u,v))\,\underbrace{\|\vec{r}_u \times \vec{r}_v\|\,du\,dv}_{\text{tiny area}}$$

To define the integral of a vector field $\vec{F}$ we will integrate the "normal component of $\vec{F}$", which is a scalar measuring the amount of $\vec{F}$ $\perp$ to the surface.



$$\text{how much} = \frac{\vec{F} \cdot (\vec{r}_u \times \vec{r}_v)}{\|\vec{r}_u \times \vec{r}_v\|}$$

So the definition of the integral is

$$\iint_{\text{surface}} \vec{F}$$

$$= \iint \underbrace{\frac{\vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)}{\|\vec{r}_u \times \vec{r}_v\|}}_{\text{scalar}} \underbrace{\|\vec{r}_u \times \vec{r}_v\| \, du\, dv}_{\text{tiny area}}$$

Some intuition:

$\vec{F}$ = velocity field of a fluid (of constant mass density)

Then $\iint_{\text{surface}} \vec{F}$ = rate of flow across the surface

= volume (mass) / area / time.

This is why $\iint_{\text{surface}} \vec{F}$ is often called a "Flux" integral

"Flux" = "Flow"

In the case of Stokes Thm, we don't think of $\nabla \times \vec{F}$ as "velocity", but the math definition is the same,

To be precise:

$\vec{r}(u,v)$ = parametrized surface
$\vec{r}(t)$ = parametrized boundary curve.

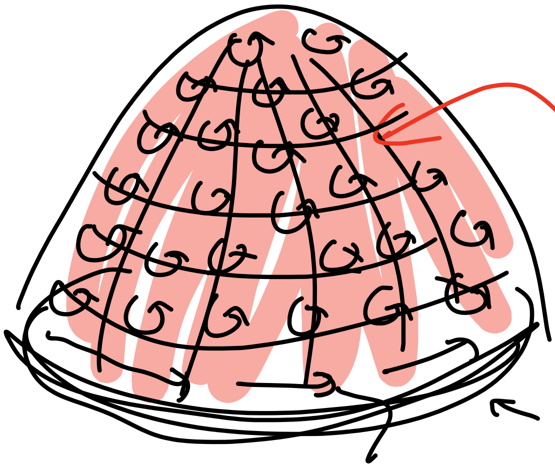
$$\iint_{\text{surface}} \nabla \times \vec{F} = \int_{\text{curve}} \vec{F}$$

$$\iint (\nabla \times \vec{F})(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

$$= \int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

Example: Check Stokes' Thm for $\vec{F}(x,y,z) = \langle -2y, 2x, x^2 z \rangle$ over the surface $z = 1 - x^2 - y^2$ for $z \geq 0$:

$u^2 = x^2 + y^2$



parabolic dome

$\vec{r}(u,v) =$

$\langle u\cos v, u\sin v, 1-u^2 \rangle$

$\vec{r}(t) = \langle \cos t, \sin t, 0 \rangle$

Summary:

$\nabla \times \vec{F} = \langle 0, -2xz, 4 \rangle$

$\nabla \times \vec{F}(\vec{r}(u,v)) = \langle 0, -2u\cos v(1-u^2), 4 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 2u^2\cos v, 2u^2\sin v, u \rangle$

$$\iint (\nabla \times \vec{F})(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

$$= \int_{v=0}^{2\pi}\int_{u=0}^{1} \left(4u - 4u^3\cos v\,(1-u^2)\right) du\, dv$$

$$= \underset{\text{computer}}{\cdots\cdots} = 4\pi$$

Now integrate along boundary curve:

$$\vec{r}(t) = \langle \cos t, \sin t, 0 \rangle$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 0 \rangle$$

$$\vec{F}(\vec{r}(t)) = \langle -2y, 2x, x^2 z \rangle$$

$$= \langle -2\sin t, 2\cos t, 0 \rangle$$

$$\int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int \langle -2\sin t, 2\cos t, 0 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$

$$= \int (+2\sin^2 t + 2\cos^2 t)\, dt$$

$$= \int_0^{2\pi} 2\, dt = 4\pi \quad \checkmark$$

This way is easier!

Why do we care?

Suppose $\nabla \times \vec{F} = \langle 0, 0, 0 \rangle$

i.e. suppose

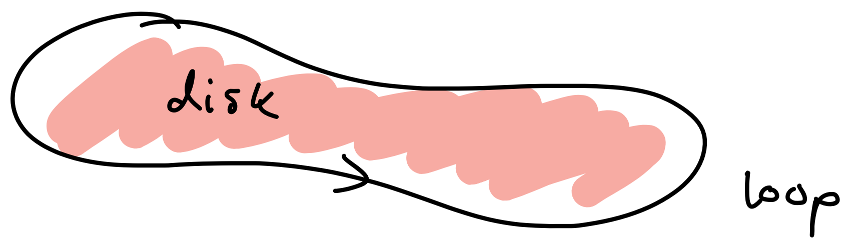
$$\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle = \langle 0,0,0 \rangle$$

i.e. suppose $\begin{cases} R_y = Q_z \\ P_z = R_x \\ Q_x = P_y \end{cases}$

Then from Stokes' Theorem we get

$$\oint_{\text{loop}} \vec{F} = 0.$$

Indeed:

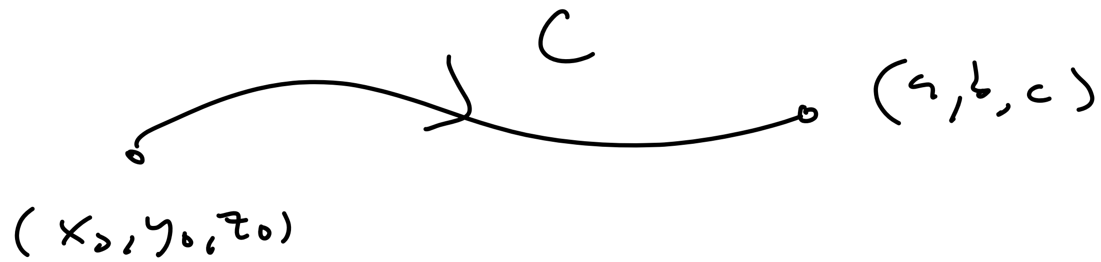


$$\oint_{\text{loop}} \vec{F} = \iint_{\text{disk}} \nabla \times \vec{F}$$

$$= \iint_{\text{disk}} \langle 0,0,0 \rangle = 0.$$

Then we can show that $\vec{F}$ has an antiderivative: $\vec{F} = \nabla f$.

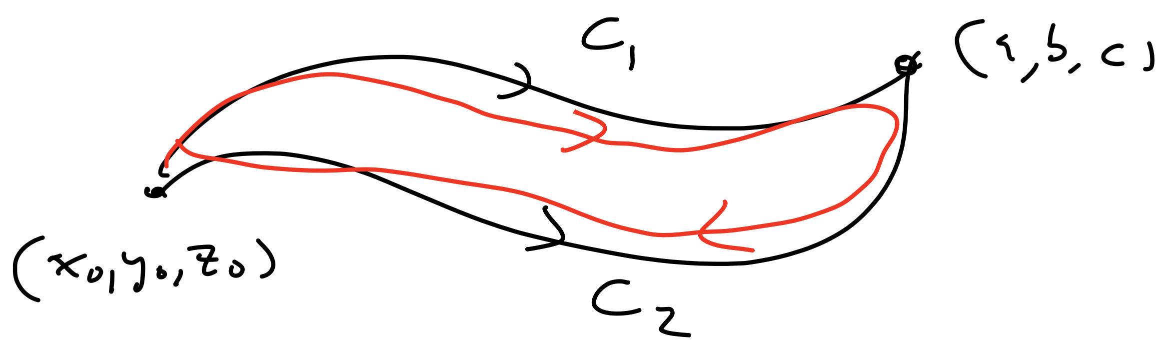
Idea: Want to define $f(a,b,c)$ for any point (a,b,c). Fix a basepoint (x_0,y_0,z_0) and choose any curve



Define $f(a,b,c) = \int_C \vec{F}$

Then it will follow from Fund Thm of line integrals that $\nabla f = \vec{F}$.

Only one problem. How do we know that the curve C is arbitrary? Take two curves:



Consider the loop "$C_1 - C_2$"

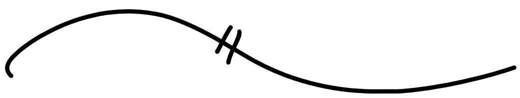
Then

$$\int_{C_1} \vec{F} - \int_{C_2} \vec{F} = \int_{C_1 - C_2} \vec{F} = 0$$

this is a loop

$$\int_{C_1} \vec{F} = \int_{C_2} \vec{F}$$

so the path really doesn't matter.



That was fancy. Let's go to 2D.

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle.$$

$$\nabla \times \vec{F} = ?$$

Makes no sense because cross product is a 3D concept.

But we can fake it:

$\vec{F} = \langle P, Q, 0 \rangle$ nothing happens in the z-direction.

$$\nabla \times \vec{F} = \langle 0, 0, \underbrace{Q_x - P_y}_{\text{this tells you about rotation in the xy-plane.}} \rangle$$

Define the 2D curl:

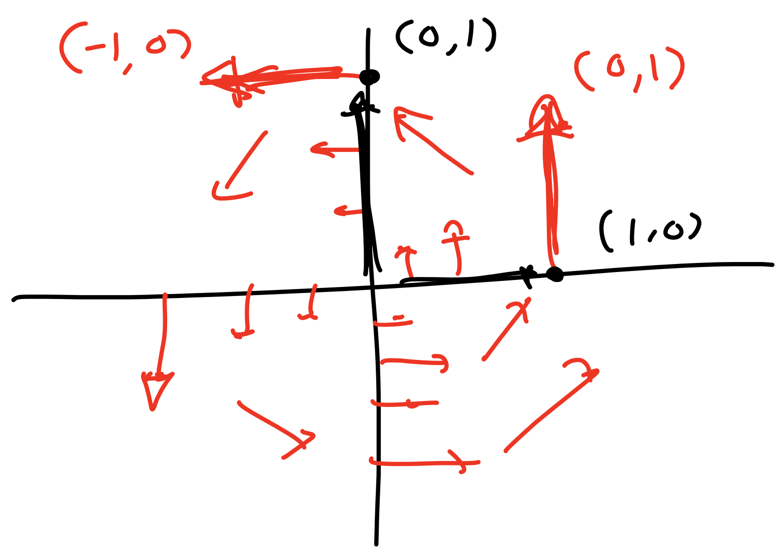
$$\operatorname{curl}(\vec{F}) = Q_x - P_y$$

$\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2 \rightsquigarrow \operatorname{curl}(\vec{F}) : \mathbb{R}^2 \to \mathbb{R}$

vector field $\longrightarrow$ scalar field.

Typical Example:

$$\vec{F} = \langle -y, x \rangle = \langle P, Q \rangle$$

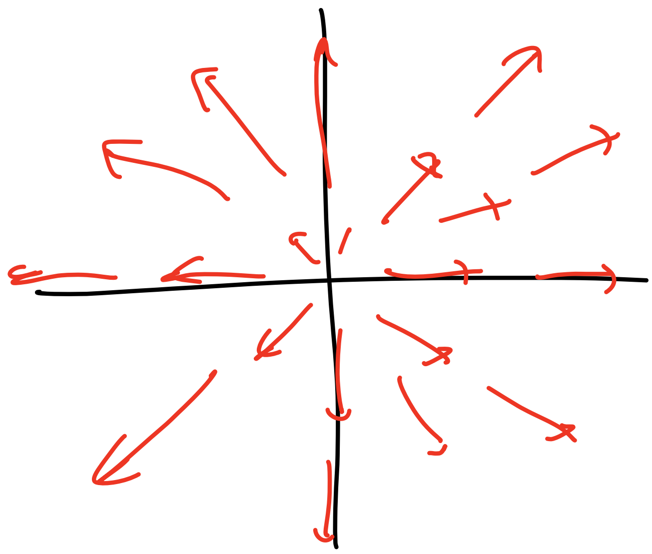


curl should be nonzero, because this field is rotating.

$$\operatorname{curl}(\vec{F}) = Q_x - P_y$$
$$= 1 - (-1) = 2 > 0$$

Indicates counterclockwise rotation.

Now consider $\vec{G} = \langle x, y \rangle = \langle P, Q \rangle$



$$\operatorname{curl}(\vec{G}) = Q_x - P_y = 0 - 0 = 0$$

This field is not rotating.

Final Project due Fri: 11:59 PM
on Blackboard.

Bonus Lecture:

Stokes Theorem (3D) $\rightsquigarrow$ Green's Theorem (2D).

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle.$$

$$\text{curl}(\vec{F}) = Q_x(x,y) - P_y(x,y).$$

[detects c.c.w. rotation]

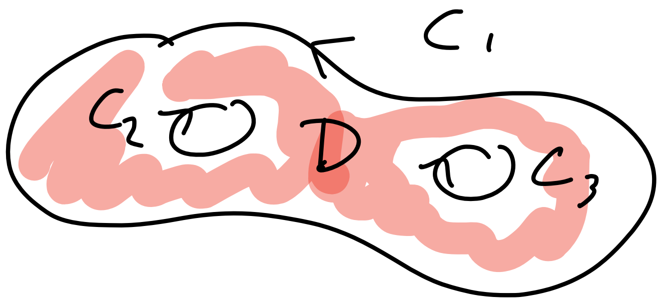
Green (special case of Stokes):

$$\iint_D \text{curl}(\vec{F}) = \int_{\partial D} \vec{F}$$

$$\iint (Q_x - P_y)\,dx\,dy = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

for parametrization $\vec{r}(t)$ of bdry curve.

The boundary curve can have multiple components:



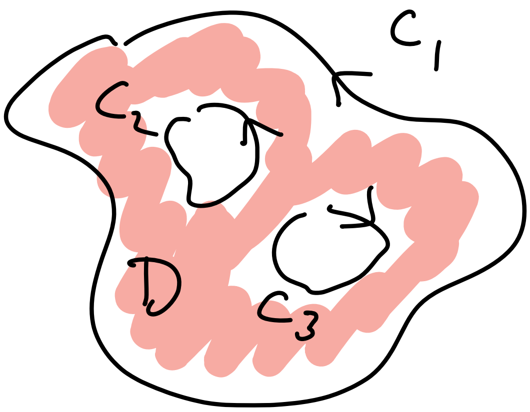
$\partial D = C_1 + C_2 + C_3$

[Rule: D is "to the left" of ∂D.]

Reason for notation:

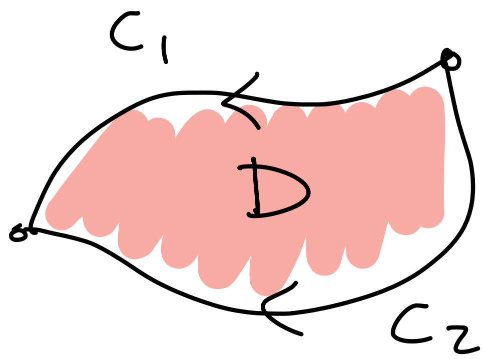
$$\int_{C_1+C_2+C_3} \vec{F} = \int_{C_1} \vec{F} + \int_{C_2} \vec{F} + \int_{C_3} \vec{F}$$

We can also reverse orientation:



$\partial D = C_1 - C_2 + C_3$

[C_2 is backwards: it has D "on the right"]



$\partial D = C_1 - C_2$

e.g. IF $\operatorname{curl}(\vec{F}) = 0$ on D. Then

$$0 = \iint_D \operatorname{curl}(\vec{F}) = \int_{C_1 - C_2} \vec{F}$$

$$= \int_{C_1} \vec{F} - \int_{C_2} \vec{F}$$

$$\Longrightarrow \int_{C_1} \vec{F} = \int_{C_2} \vec{F}.$$

Summary:

$\operatorname{curl}(\vec{F}) = 0$

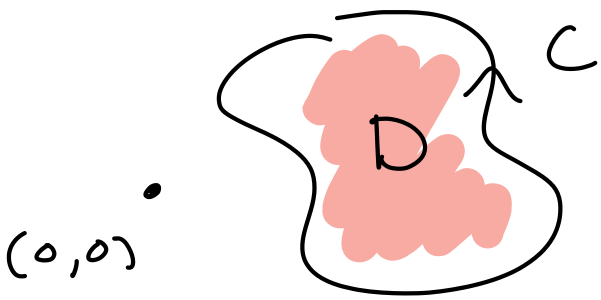
$\Longrightarrow \int_C \vec{F}$ only depends on endpoints of C, not the shape.

Example: $\vec{F}(x,y) = \frac{1}{x^2+y^2}\langle -y, x\rangle$.

$P = -y/x^2+y^2 \rightsquigarrow P_y = y^2-x^2/(x^2+y^2)^2$

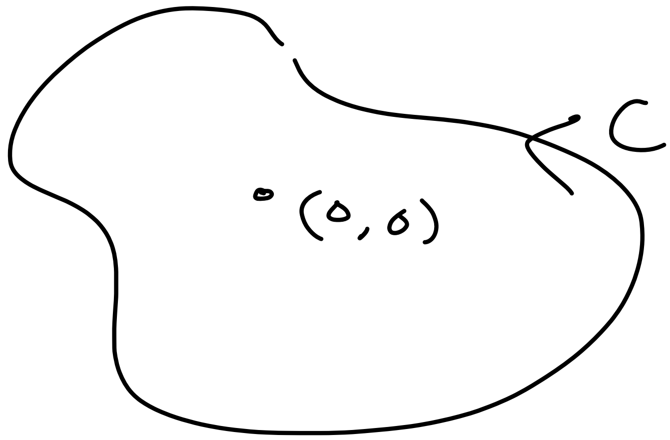
$Q = x/x^2+y^2 \rightsquigarrow Q_x = y^2-x^2/(x^2+y^2)^2$.

So $\operatorname{curl}(\vec{F}) = Q_x - P_y = 0$, when it is defined. It's not defined at $(0,0)$. If a loop C does not contain $(0,0)$



then $\int_C \vec{F} = \iint_D \operatorname{curl}(\vec{F}) = \iint_D 0 = 0$.

What about a loop that contains $(0,0)$?



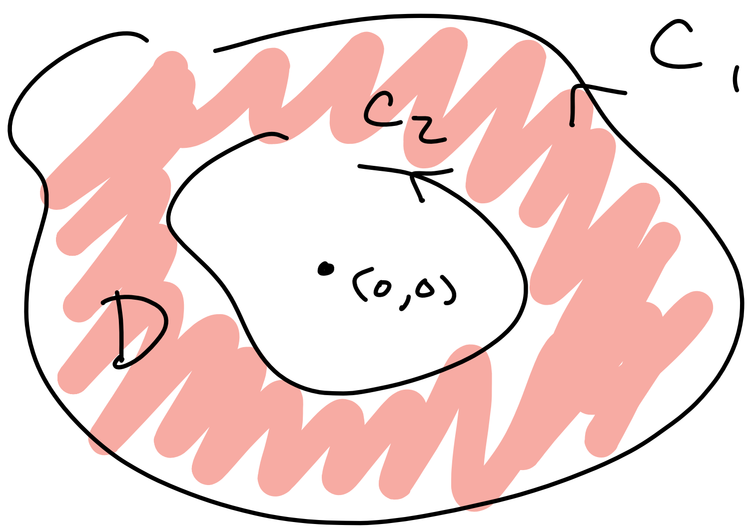
$$\int_C \vec{F} = ?$$

Claim: $\int_C \vec{F} = 2\pi$,

independent of the shape of C.

Proof has 2 steps:

(1) Any two such loops have the same integral:



$$\partial D = C_1 - C_2$$

$$0 = \iint_D \operatorname{curl}(\vec{F}) = \int_{C_1 - C_2} \vec{F}$$

$$= \int_{C_1} \vec{F} - \int_{C_2} \vec{F}$$

$$\Longrightarrow \int_{C_1} \vec{F} = \int_{C_2} \vec{F}.$$

(2) So pick the easiest curve.

$$\vec{r}(t) = \langle \cos t, \sin t \rangle.$$

$$\int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{2\pi} \left\langle \frac{-\sin t}{\underbrace{\cancel{\cos^2 t + \sin^2 t}}_{1}}, \frac{\cos t}{\underbrace{\cancel{\cos^2 t + \sin^2 t}}_{1}} \right\rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int (\sin^2 t + \cos^2 t)\, dt$$

$$= \int_0^{2\pi} 1 \, dt = 2\pi \quad \checkmark$$

Summary: $\vec{F}(x,y) = \dfrac{1}{x^2+y^2}\langle -y, x\rangle.$

$$\oint_C \vec{F} = \begin{cases} 0 & C \text{ does not contain } (0,0) \\ 2\pi & C \text{ goes around } (0,0) \text{ once in c.c.w. direction} \\ 2\pi k & C \text{ goes around } (0,0)\ k \text{ times in c.c.w. direction.} \end{cases}$$

Flux form of Green's Theorem.

Let $\vec{F} = \langle P, Q\rangle$

$\vec{G} = \langle u, v\rangle = \langle -Q, P\rangle.$

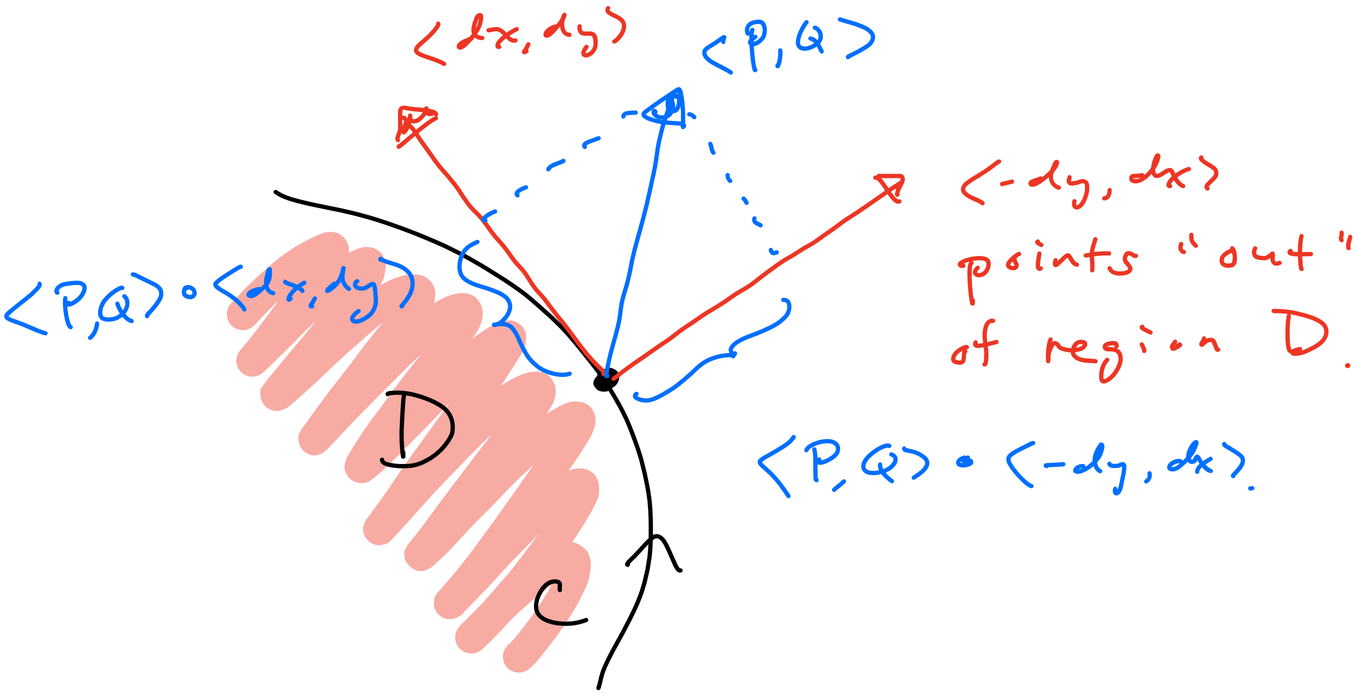
Apply Green to $\vec{G}$.

$$\iint_D (V_x - U_y)\,dx\,dy = \int_{\partial D} \langle u, v\rangle \cdot \langle dx, dy\rangle.$$

$$\iint (P_x + Q_y)\,dx\,dy = \int \langle -Q, P\rangle \cdot \langle dx, dy\rangle$$

$$= \int \langle P, Q\rangle \cdot \langle -dy, dx\rangle.$$

What?



so $\displaystyle\int_C \langle P,Q \rangle \cdot \langle -dy, dx \rangle$

measures how much the vector field $\langle P, Q \rangle$ points "out of" the region D. Called "Flux".

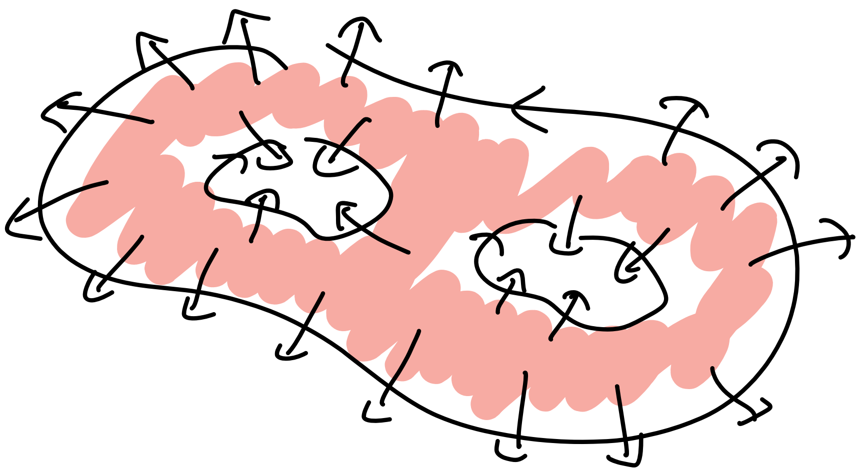
Cleaner Notation:

$$\underbrace{\iint_D (Q_x - P_y)\,dx\,dy}_{\text{amount of curling in } D} = \underbrace{\int_{\partial D} \vec{F} \cdot \vec{T}}_{\text{amount } \vec{F} \text{ points along } \partial D.}$$

little tangent vector

$$\underbrace{\iint_D (P_x + Q_y)\,dx\,dy}_{\text{amount that } \vec{F} \text{ expands inside } D} = \underbrace{\int_{\partial D} \vec{F} \cdot \vec{N}}_{\text{amount that } \vec{F} \text{ flows across } \partial D.}$$

little normal vector.



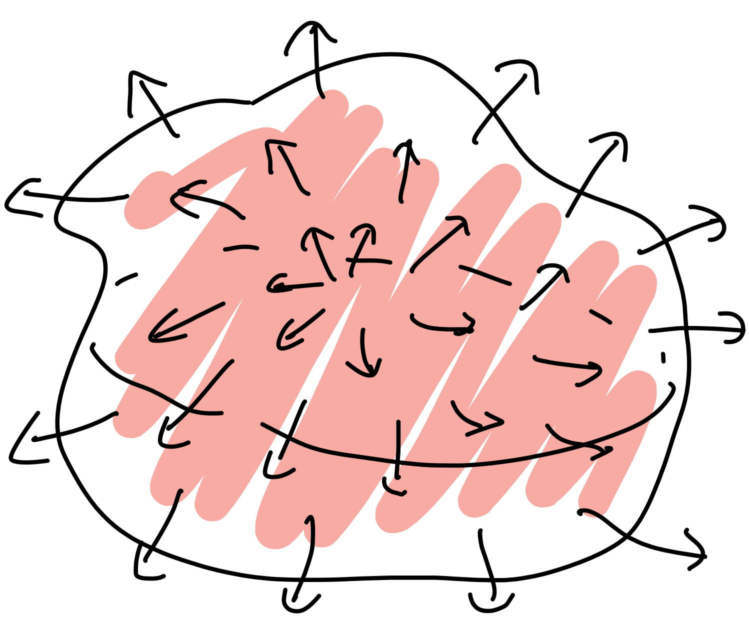
Moving from 2D to 3D: Green's Theorem becomes two different theorems.

$$\begin{tikzcd} & \text{Green} \arrow[dl, "\text{circulation}"'] \arrow[dr, "\text{flux}"] & \\ \text{Stokes} & & \text{Gauss (Divergence Theorem)} \end{tikzcd}$$

We've seen Stokes.

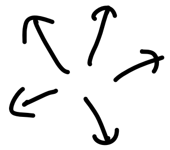
Now: Gauss' Theorem

$$\underbrace{\iiint_V \nabla \cdot \vec{F}}_{\text{expansion of } \vec{F} \text{ in a volume } V} = \underbrace{\iint_{\partial V} \vec{F} \cdot \vec{N}}_{\text{Flow of } \vec{F} \text{ across boundary surface } \partial V.}$$

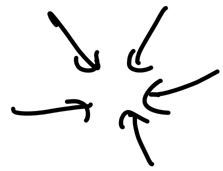


$$\nabla \cdot \langle P, Q, R \rangle = P_x + Q_y + R_y$$

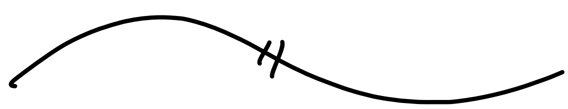
Gauss tells us what this strange formula has to do with "expansion/contraction".



$\nabla \cdot \vec{F} > 0$



$\nabla \cdot \vec{F} < 0$



Application to Gravity.

Let $\vec{F}(x,y,z)$ be the gravitational force acting on a particle of mass m at point (x,y,z), due to some mass distribution $\rho(x,y,z)$.

assume gravitational constant = 1

Gauss' Law:

$$\nabla \cdot \vec{F} = -4\pi m \, \rho(x,y,z)$$

This is equivalent to (and more useful than) Newton's universal gravitation. It is particularly useful when dealing with a spherically symmetric density ρ.

Suppose

$$M = \text{total mass} = \iiint \rho \, dV$$

Suppose

$$\vec{F}(\vec{r}) = \underbrace{F(r)}_{\text{some scalar function of } r = \|\vec{r}\|} \frac{\vec{r}}{\|\vec{r}\|}$$

i.e. the force is the same in all directions, only depends on the distance from $(0,0,0)$.

Then Divergence Theorem says

$$\underset{\substack{\text{ball} \\ \text{radius } r}}{\iiint} \nabla \cdot \vec{F} = \underset{\substack{\text{sphere} \\ \text{radius } r}}{\iint} \underbrace{\vec{F} \cdot \vec{N}}$$

we can use $\vec{N} = \dfrac{\vec{r}}{\|\vec{r}\|}$

this is the component of $\vec{F}$ normal to surface of the sphere.

$$\iiint -4\pi m \rho = \iint F(r)$$

$$-4\pi m M_r = F(r)\, 4\pi r^2$$

how much mass inside ball of radius r.

surface area of sphere

Three interesting Examples:

- Point particle at $(0,0,0)$ mass M.

$F(r) = -Mm/r^2$ (Newton).

- Solid sphere radius R.

$$F(r) = \begin{cases} -Mm/r^2 & r \geq R. \\ -\dfrac{Mm}{R^3}\, r & r < R \end{cases}$$

- Empty shell radius R with all mass M on its boundary

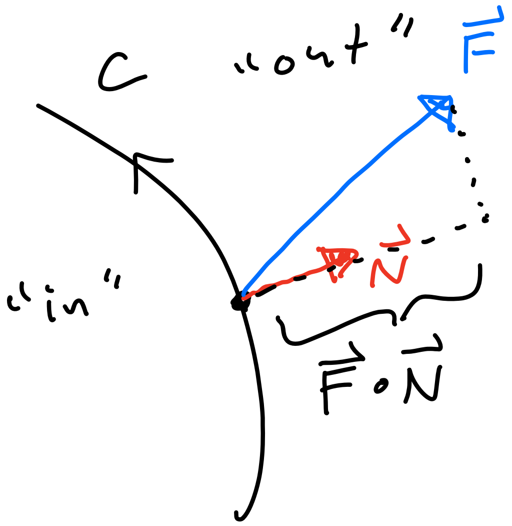
$$F(r) = \begin{cases} -Mm/r^2 & r \geq R \\ 0 & r < R. \end{cases}$$

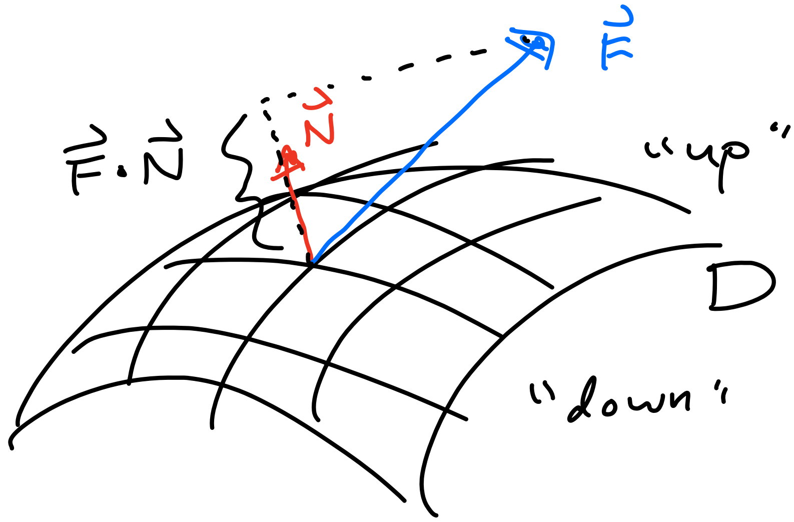
Inside a massive spherical shell you fell <u>no</u> gravity.

Called "Newton's shell theorem", proved by him using complicated argument.

Gauss' Law & Divergence Thm make it "almost obvious" ;)

How to compute "Flux" across curves in $\mathbb{R}^2$ & surfaces in $\mathbb{R}^3$:





$$\int_C \vec{F}\cdot\vec{N}\,ds \qquad \iint_D \vec{F}\cdot\vec{N}\,dA$$

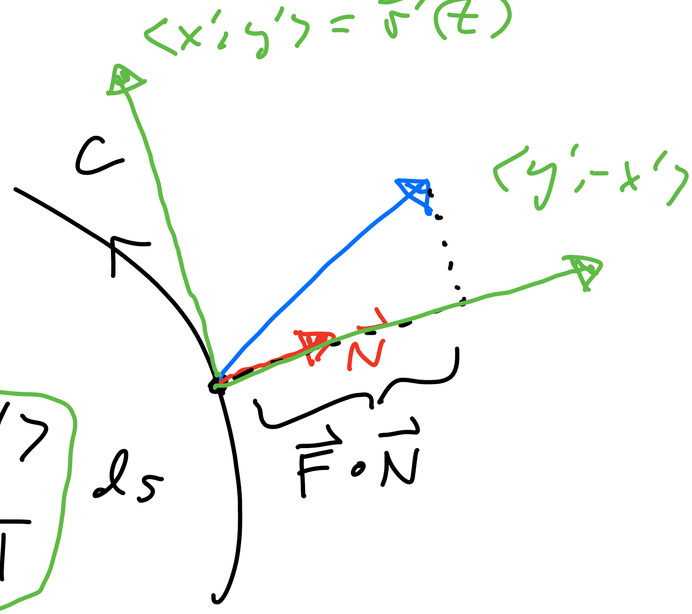
To compute: choose parametrizations

$C: \ \vec{r}(t) = \langle x(t), y(t)\rangle$

$D: \ \vec{r}(u,v)$

$$\int_C \vec{F}\cdot\vec{N}\,ds$$

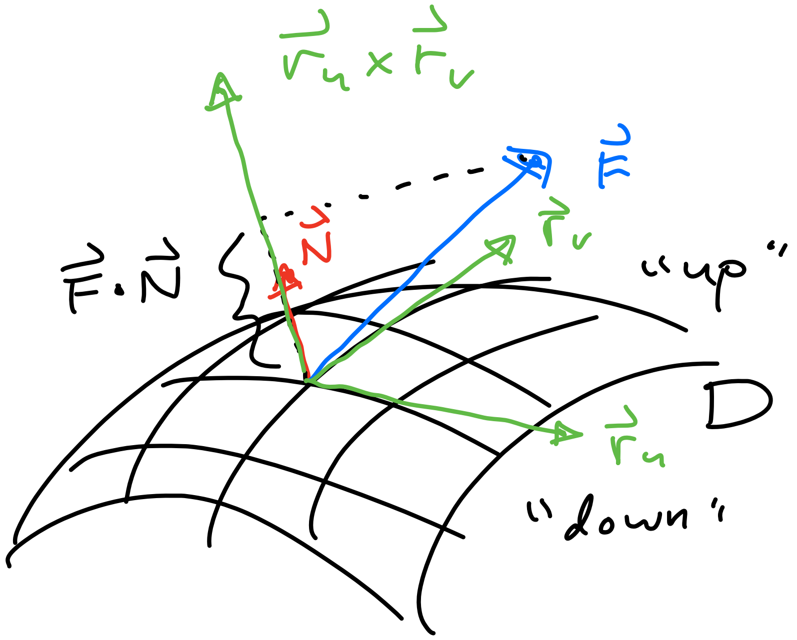
$$= \int \vec{F}(\vec{r}(t))\cdot \boxed{\frac{\langle y', -x'\rangle}{\|\vec{r}\,'(t)\|}}\,ds$$



$$= \int \vec{F}(\vec{r}(t)) \cdot \frac{\langle y', -x' \rangle}{\cancel{\|\vec{r}'(t)\|}} \cancel{\|\vec{r}'(t)\|}\, dt$$

$$= \int \vec{F}(x(t), y(t)) \cdot \langle y'(t), -x'(t) \rangle \, dt$$

$$\iint_D \vec{F} \cdot \vec{N} \, dA$$



$$= \iint \vec{F}(\vec{r}(u,v)) \cdot \frac{\vec{r}_u \times \vec{r}_v}{\cancel{\|\vec{r}_u \times \vec{r}_v\|}} \cancel{\|\vec{r}_u \times \vec{r}_v\|} \, du\, dv$$

$$= \iint \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \, du\, dv$$

The Fundamental Theorems:

- Given 2D region D in $\mathbb{R}^2$ and a vector field. $\vec{F} = \langle P, Q \rangle$

Green:

$$\iint_D \underbrace{\text{curl}(\vec{F})}_{Q_x - P_y} \, dA = \int_{\partial D} \vec{F} \cdot \vec{T} \, ds$$

line integral

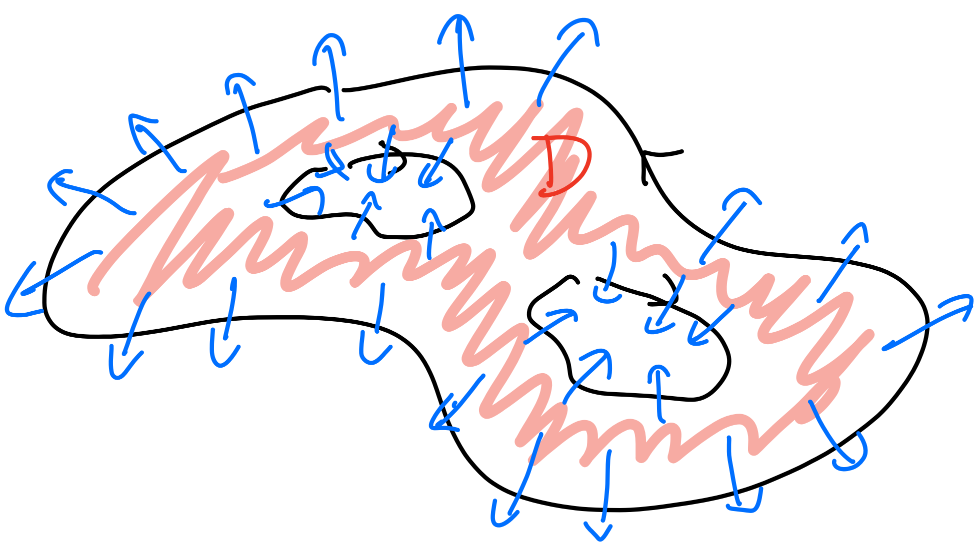
Green (Flux form):

$$\boxed{\iint_D \underbrace{\text{div}(\vec{F})}_{P_x + Q_y} \, dA} = \boxed{\int_{\partial D} \vec{F} \cdot \vec{N} \, ds}$$

expansion of $\vec{F}$ inside D

Flow of $\vec{F}$ across boundary

"Proof":



Think: $\vec{F}$ = velocity field of a fluid. Then $\operatorname{div}(\vec{F})$ = expansion.

In fluid mechanics we often assume $\nabla \cdot \vec{F} = 0$ (incompressible).
$\operatorname{div}(\vec{F}) = 0$

Move into 3D:

Green $\longrightarrow$ Stokes

$$\iint_D (\nabla \times \vec{F}) \cdot \vec{N} \, dA = \int_{\partial D} \vec{F} \cdot \vec{T} \, ds.$$

"Proof": $\partial D = C_1 + C_2 + C_3$

NEW :

Flux Form of Green $\longrightarrow$ Divergence Thm (Gauss' Theorem).

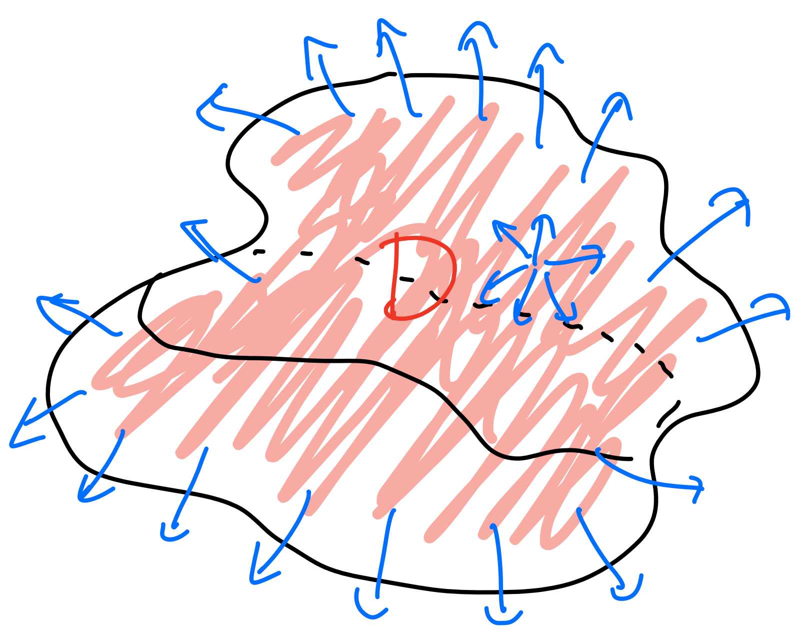
Let D solid 3D region in $\mathbb{R}^3$

Let ∂D be oriented 2D boundary surface.

$$\iiint_D \nabla \cdot \vec{F}\, dV = \iint_{\partial D} \vec{F} \cdot \vec{N}\, dA$$

"Proof": Expansion of fluid "inside" D = Flow of $\vec{F}$ "out" of region :

"Conservation of Mass"



Original Application : Gauss.

Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be gravitational force due to some mass density distribution $\rho : \mathbb{R}^3 \longrightarrow \mathbb{R}$, acting on a point particle of mass 1 sitting at point (x, y, z).

assume $G = 1$

Gauss :

$$\boxed{\nabla \cdot \vec{F} = -4\pi \rho}$$

Implies Newton's law of gravitation, but Gauss is more useful!

General computations are "impossible".

Nice case : Distribution $\rho(x,y,z)$ is spherically symmetric :

$$\vec{F}(\vec{r}) = F_r \frac{\vec{r}}{\|\vec{r}\|}$$

write $M_r = \iiint\limits_{\substack{\text{Ball of} \\ \text{radius } r}} \rho \, dV$

$=$ total mass inside ball radius r.

Three examples:

- point particle mass M $\qquad M_r = M \quad r > 0.$

- solid sphere radius R $\qquad M_r = \begin{cases} M & r > R \\ \frac{r^3}{R^3} M & r < R \end{cases}$

$$M_r = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} \cdot M = \frac{r^3}{R^3} M$$

ratio that we want

volume of full sphere

- hollow spherical shell $\qquad M_r = \begin{cases} M & r > R \\ 0 & r < R \end{cases}$

In general: Gauss says

$$\iiint_{\substack{\text{ball} \\ \text{radius } r}} \nabla \cdot \vec{F} = \iint_{\substack{\partial \text{ ball} \\ \text{radius } r}} \vec{F} \cdot \vec{N} \, dA$$

$$\iiint -4\pi \rho = \iint F_r \underbrace{\frac{\vec{r}}{\|\vec{r}\|} \cdot \frac{\vec{r}}{\|\vec{r}\|}}_{1} \, dA$$

$$-4\pi \underbrace{\iiint \rho}_{M_r} = F_r \underbrace{\iint 1 \, dA}_{4\pi r^2}$$

surface area sphere radius r

$$-\cancel{4\pi} M_r = F_r \cancel{4\pi} r^2$$

$$\boxed{F_r = -M_r / r^2}$$

True for any spherically symm. mass distribution.

- Point mass:

$$F_r = -M/r^2 \quad \text{(Newton)}$$

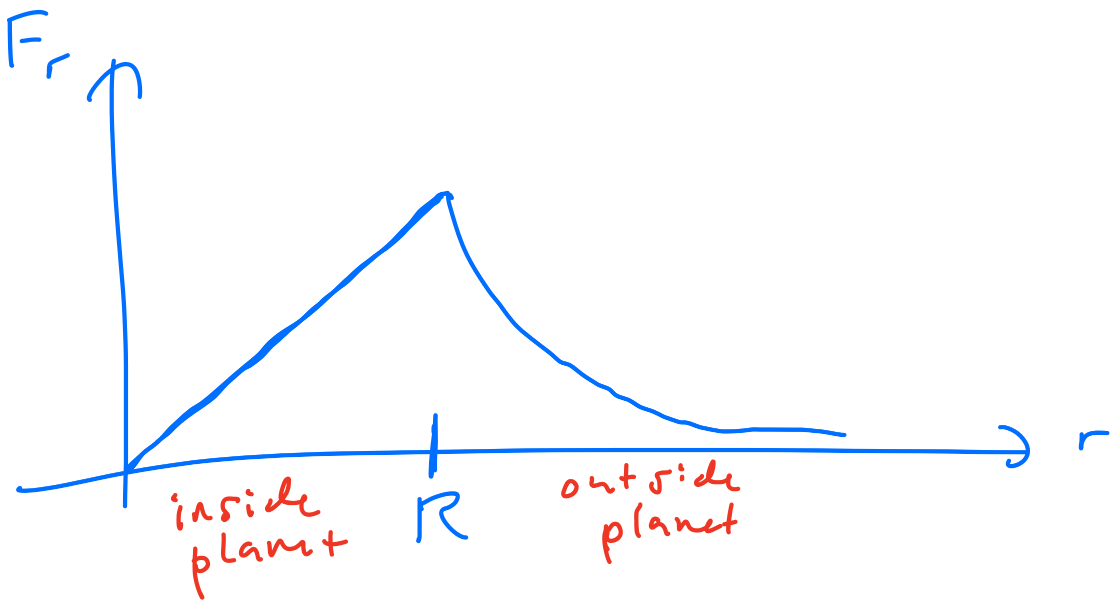
- Solid sphere (planet) radius R

$$F_r = -M/r^2 \qquad r > 0$$

(if you are outside the planet, it acts on you like a point particle)

$$F_r = -\frac{r^3}{R^3} M / r^2$$

$$= -\frac{M}{R^3} \cdot r$$ scales like r.



- Empty spherical shell:

$$F_r = \begin{cases} -M/r^2 & r > R \\ \boxed{0 \qquad r < R} \end{cases}$$

very interesting!

You feel NO GRAVITY inside of a massive spherical shell.