Bonus Lecture : The physics behind the fundamental theorems of vector calculus.

Recall Stokes' Theorem

$$\iint_D (\nabla \times \vec{F}) \cdot \vec{N} \, dA = \oint_{\partial D} \vec{F} \cdot T \, ds$$

and the Divergence Theorem

$$\iiint_E (\nabla \cdot \vec{F}) \, dV = \oiint_{\partial E} \vec{F} \cdot \vec{N} \, dA$$

[Jargon : $\oint$ and $\oiint$ indicate integration over closed curves & surfaces, e.g. circles & surfaces of spheres.]

Physics : Scalar & Vector fields can change with time.

- $f(x,y,z,t) = f(\vec{x},t)$ could be a changing temperature distribution.
- $\vec{F}(x,y,z,t) = \vec{F}(\vec{x},t)$ could be a changing magnetic field.

This allows us to talk about time derivatives as well as spatial derivatives. For example:

$$f_t \quad \text{vs.} \quad \nabla f$$

$$\vec{F}_t \quad \text{vs.} \quad \nabla \times \vec{F} \quad \text{or} \quad \nabla \cdot \vec{F}$$

And you can mix them:

$$\nabla(f_t) = (\nabla f)_t$$

$$\nabla \times (\vec{F}_t) = (\nabla \times \vec{F})_t$$

$$\nabla \cdot (\vec{F}_t) = (\nabla \cdot \vec{F})_t$$

The Heat (or Diffusion) Equation:

Let $f(x,y,z,t)$ be a temperature distribution in a "uniform medium".

The "heat energy content" of a 3D region E is defined as the volume integral of the temperature:

$$\text{heat} = \iiint \text{temperature}\, dV$$

$$= \iiint_E f\, dV$$

Fourier's Law says that the "heat flow" is proportional to the gradient of the temperature

$$\frac{d}{dt}(\text{heat}) \approx -\nabla(\text{temperature})$$

[Heat flow tries to decrease the temperature gradient.]

Combine that with our definition of "heat" to get

$$\frac{d}{dt} \iiint_E f \, dV = \oiint_{\partial E} -\nabla f \cdot \vec{N} \, dA$$

change in heat content = heat flow across the boundary

"conservation of energy"

Finally, we combine this with the Divergence Theorem:

$$\frac{d}{dt} \iiint_E f \, dV = - \oiint_{\partial E} (\nabla f) \, dA$$

$$\iiint_E f_t \, dV = - \iiint_E (\nabla \cdot (\nabla f)) \, dV$$

Since this is true for any 3D region E, it must be true that the integrands are equal:

$$f_t = -\nabla\cdot(\nabla f)$$

$$= -\langle \partial_x, \partial_y, \partial_z\rangle \cdot \langle f_x, f_y, f_z\rangle$$

$$= -(f_{xx} + f_{yy} + f_{zz})$$

[Jargon: The divergence of the gradient is called the "Laplacian operator":

$$\text{"}\nabla^2 f\text{"} = \text{"}(\nabla\cdot\nabla) f\text{"}$$

$$= \nabla\cdot(\nabla f)$$

$$= f_{xx} + f_{yy} + f_{zz}.$$

This is a 3D version of the "second derivative".]

Thus we can write

$$\boxed{f_t = -\nabla^2 f}$$

This is called the "heat equation".

More generally, it applies to any kind of "gradient diffusion".

The Wave Equation:

Let $f(x, y, t)$ be a small vertical displacement of a horizontal membrane in the xy-plane (e.g. a drum).

Hooke's Law:

$$\text{elastic force} \approx -\nabla f$$

The gradient causes a "restoring force"

[Before, $-\nabla f$ was a first deriv of t. Now it's a second derivative of t.]

Conservation of Momentum:

The internal forces in any region cancel, leaving only the forces acting across the boundary:

$$\begin{matrix}\text{total sum of} \\ \text{forces on a} \\ \text{region } D\end{matrix} = \oint_{\partial D} \text{force} \cdot \vec{N}\, ds$$

If the drum head has uniform mass density $\rho = 1$:

$$\underbrace{1 \cdot f_{tt}\, dx\, dy}_{\text{mass} \cdot \text{acceleration!}} = \begin{matrix}\text{force acting on a} \\ \text{tiny piece of drum}\end{matrix}$$

Combining everything:

$$\text{total force} = \oint \text{force} \cdot \vec{N}\, ds$$

$$\iint_D f_{tt}\, dA = \oint_{\partial D} \nabla f \cdot \vec{N}\, ds$$

flux form of Green

$$= \iint_D \nabla \cdot (\nabla f)\, dA$$

Since this is true for <u>any</u> region D, the integrands must be equal:

$$f_{tt} = \nabla \cdot (\nabla f)$$

$$\boxed{f_{tt} = \nabla^2 f}$$

This is called the "wave equation".

Idea: The slope of the drum head causes a restoring force to make it flat. But, unlike heat flow, the restoring force can "overshoot", causing wave propagation. The <u>same</u> equation holds for acoustic waves in 3D, but now $f(x,y,z,t)$ is "pressure", not "height".

Gauss' Law for Gravity:

Consider a mass distribution $\rho(x,y,z)$. This distribution causes a gravitational force $\vec{g}(x,y,z)$ acting on a unit mass particle at the point (x,y,z).

Gauss' Law:

$$\nabla \cdot \vec{g} = -4\pi \rho$$

Integrate both sides over a 3D region E and use the Divergence Thm:

$$\iiint_E \nabla \cdot \vec{g} \, dV = -4\pi \iiint_E \rho \, dV$$

$$\oiint_{\partial E} \vec{g} \cdot \vec{N} \, dA = -4\pi M,$$

where M is the total mass inside the region E.

This is particularly useful when the mass distribution has some symmetry.

Example: Let $\rho(x,y,z)$ have spherical symmetry around $(0,0,0)$ (e.g. a spherical planet). Then the force of gravity points directly towards the origin:

$$\vec{g}(\vec{r}) = -\|\vec{g}(\vec{r})\| \frac{\vec{r}}{\|\vec{r}\|}$$

Write $r = \|\vec{r}\|$ & $g(r) = \|\vec{g}(\vec{r})\|$, which only depends on the distance r of our "test mass" from $(0,0,0)$.

If E is the ball of radius r then Gauss' Law says that

$$\iiint_{\substack{\text{ball of}\\ \text{radius } r}} -4\pi\rho \, dV = \oiint_{\substack{\text{sphere}\\ \text{of radius } r}} g(r) \cancel{\frac{\vec{r}}{r} \cdot \vec{N}}^{\,1} \, dA$$

$$\underbrace{-4\pi M}_{\substack{\text{total mass}\\ \text{inside the ball}}} = g(r) \underbrace{\iint dA}_{\substack{\text{surface area}\\ \text{of the sphere}}}$$

$$-4\pi M = g(r) \cdot 4\pi r^2$$

$$\boxed{g(r) = -M/r^2}$$

This implies Newton's Law of Universal Gravitation. Indeed, if $\rho(x,y,z)$ is concentrated in a point of mass M at $(0,0,0)$, then a point of mass m with distance r from $(0,0,0)$ feels the force

$$m g(r) = -Mm/r^2 \qquad \checkmark$$

But Gauss' Law is much more general. It says that the force $g(r) = -M/r^2$ doesn't depend on the exact shape of $\rho(x,y,z)$; only on the fact that ρ has spherical symmetry & the total mass inside the ball of radius r is M.

Consequences:

- Let ρ be a uniform sphere of radius R & total mass M (hence $\rho = M/(\frac{4}{3}\pi R^3)$), then

$$g(r) = \begin{cases} -M/r^2 & \text{if } r \geq R \\ -\frac{M}{R^3} \cdot r & \text{if } 0 \leq r \leq R \end{cases}$$

Inside the planet, $g(r) \sim r$.
Outside the planet, $g(r) \sim 1/r^2$ as though the planet were a point particle.

- Let ρ be an empty spherical shell of radius R & mass M. Then

$$g(r) = \begin{cases} -M/r^2 & \text{if } r \geq R \\ 0 & \text{if } 0 \leq r \leq R \end{cases}$$

From outside the empty shell is indistinguishable from a uniform sphere or a point particle. But inside the shell you feel no gravity at all. Pretty Amazing!

This result is called "Newton's Shell Theorem". He originally proved it with tricky geometric arguments. The powerful tool of Gauss' Law makes it

"almost obvious"! :)

Electromagnetism :

Let $\rho(x,y,z)$ be a distribution of "electric charge" (whatever that is).
This distribution causes a force $\vec{E}(x,y,z)$ acting on unit charge particle at the point (x,y,z).

Gauss' Law for Electricity :

$$\nabla \cdot E = \rho$$

I will ignore constants

[This tells us that the electric force is very much like gravity; except that it is repulsive. Newton's Law of gravitation $g(r) = -Mm/r^2$ becomes Coulomb's Law $E(r) = Qq/r^2$.]

But there is also a "magnetic field" $\vec{B}$ that is quite different:

$$\nabla \cdot \vec{B} = 0$$

There is no such thing as "magnetic charge" (as far as we know).
So $\vec{B}$ doesn't act anything like gravity. But the magnetic field does cause forces on electric charge.

Faraday's Law:

A changing magnetic field causes electric charge to move in circles:

$$\nabla \times \vec{E} = -\vec{B}_t$$

Conversely, charge moving in circles causes the magnetic field to change. That's rather strange.

And there is also a kind of symmetry between $\vec{E}$ & $\vec{B}$.

Ampère / Maxwell Law:

Moving electric charge (or a changing electric field) causes the magnetic field to go in circles:

$$\nabla \times \vec{B} = \vec{E}_t + \vec{J},$$

where $\vec{J}$ is the "current density" (electric current per unit area).

In the absence of charge (i.e., in a vacuum) the fields $\vec{E}$ & $\vec{B}$ become almost perfectly symmetric.

$$\begin{cases} \nabla \cdot \vec{E} = 0 \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = -\vec{B}_t \\ \nabla \times \vec{B} = +\vec{E}_t \end{cases}$$

Maxwell played around with these equations and made a huge discovery.

Take the curl of the curl:

$$\nabla \times (\cancelto{-\vec{B}_t}{\nabla \times \vec{E}}) = \nabla(\cancelto{0}{\nabla \cdot \vec{E}}) - \nabla \cdot (\nabla \vec{E})$$

$$\nabla \times (-\vec{B}_t) = 0 - \nabla^2 \vec{E}$$

$$-(\cancelto{\vec{E}_t}{\nabla \times \vec{B}})_t = -\nabla^2 \vec{E}$$

$$\boxed{\vec{E}_{tt} = \nabla^2 \vec{E}}$$

And a similar argument shows that

$$\boxed{\vec{B}_{tt} = \nabla^2 \vec{B}}$$

These look like wave equations!

What is going on?

What "material" is "waving"?

By including the relevant constants Maxwell observed that these "waves" seem to travel at the speed of light.

Maxwell had discovered by accident that light is an example of "electro-magnetic radiation".