Math 310 — Fall 2023
Homework 2 — Drew Armstrong

Problem 1. A Line in Space. Consider the line in $\mathbb{R}^3$ passing through the two points

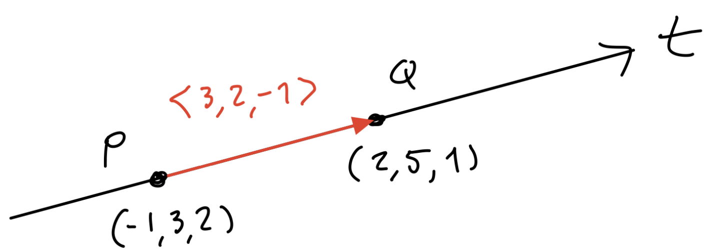
$$P = (-1, 3, 2) \quad \text{and} \quad Q = (2, 5, 1).$$

(a) Express this line in parametric form $\mathbf{r}(t) = (x_0 + ta, y_0 + tb, z_0 + tc)$.
(b) Find the equations of two planes in $\mathbb{R}^3$ whose intersection is this line. [Hint: There are infinitely many solutions. One solution uses the symmetric equations.]

(a): We can take $(x_0, y_0, z_0) = P = (-1, 3, 2)$ and $\langle a, b, c\rangle = \vec{PQ} = Q - P = \langle 3, 2, -1\rangle$ to get

$$\mathbf{r}(t) = (-1 + 3t, 3 + 2t, 2 - t).$$

Here is a picture:



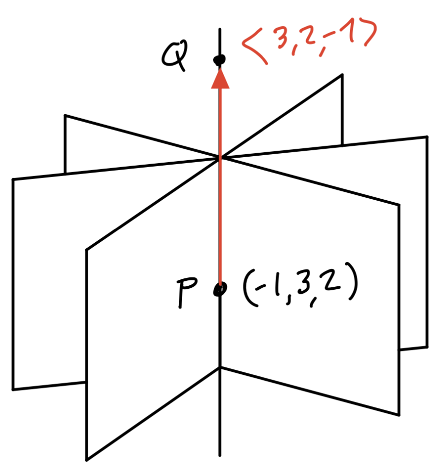
(b): A general point on the line satisfies $(x, y, z) = (-1 + 3t, 3 + 2t, 2 - t)$ for some t. We can eliminate t to obtain the "symmetric equations" of the line:

$$t = \frac{x+1}{3} = \frac{y-3}{2} = \frac{z-2}{-1}.$$

These equations tells us the line is the intersection of three planes:

$$\frac{x+1}{3} = \frac{y-3}{2} \quad \text{and} \quad \frac{x+1}{3} = \frac{z-2}{-1} \quad \text{and} \quad \frac{y-3}{2} = \frac{z-2}{-1}.$$

Here is a picture:



Actually, the third plane is redundant so we can pick any two of these planes. [More generally we can just pick any two planes that contain the line. There are infinitely many valid choices.]

Problem 2. A Plane in Space. Consider the plane in $\mathbb{R}^3$ passing through the three points

$$P = (-1, 3, 2), \quad Q = (2, 5, 1), \quad R = (0, 2, 4).$$

(a) Find a vector that is perpendicular to this plane.
(b) Find the equation of the plane.

(a): We can find a normal vector by taking the cross product of any two vectors in the plane. For example, we can take $\vec{PR} = R - P = \langle 1, -1, 2\rangle$ and $\vec{PQ} = Q - P = \langle 3, 2, -1\rangle$ to get

$$\begin{aligned}
\vec{PR} \times \vec{PQ} &= \langle 1, -1, 2\rangle \times \langle 3, 2, -1\rangle \\
&= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 3 & 2 & -1 \end{pmatrix} \\
&= \mathbf{i} \det \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} - \mathbf{j} \det \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} + \mathbf{k} \det \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} \\
&= \mathbf{i}(1 - 4) - \mathbf{j}(-1 - 6) + \mathbf{k}(2 + 3) \\
&= -3\mathbf{i} + 7\mathbf{j} + 5\mathbf{k} \\
&= \langle -3, 7, 5\rangle.
\end{aligned}$$

(b): The plane that contains the point (x_0, y_0, z_0) and is perpendicular to the vector $\langle a, b, c\rangle$ has the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

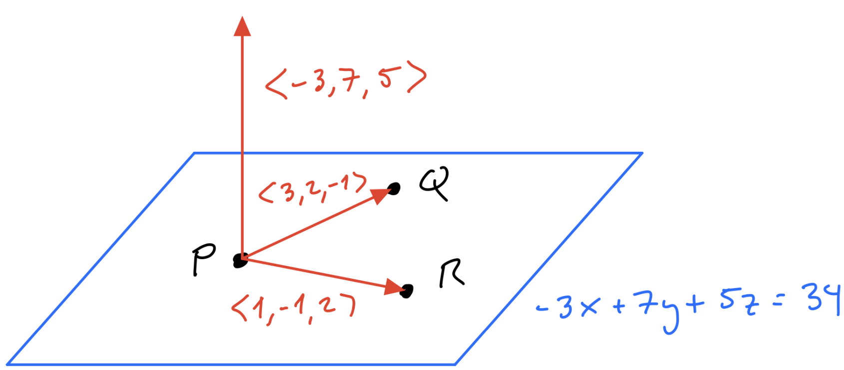
In our case we can take $(x_0, y_0, z_0) = P = (-1, 3, 2)$ and $\langle a, b, c\rangle = \langle -3, 7, 5\rangle$ to get

$$\begin{aligned}
-3(x + 1) + 7(y - 3) + 5(z - 2) &= 0 \\
-3x + 7y + 5z - 3 - 21 - 10 &= 0 \\
-3x + 7y + 5z &= 34.
\end{aligned}$$

Finally, let's check that this plane contains the three given points:

$$\begin{aligned}
-3(-1) + 7(3) + 5(2) &= 3 + 21 + 10 = 34, \\
-3(2) + 7(5) + 5(1) &= -6 + 35 + 5 = 34, \\
-3(0) + 7(2) + 5(4) &= 0 + 14 + 20 = 34.
\end{aligned}$$

It works. Here is a picture:



Problem 3. Intersection of Two Planes. Consider the following two planes in $\mathbb{R}^3$:

$$\begin{matrix} (1) \\ (2) \end{matrix} \begin{cases} x + y + 2z = 1, \\ x - y + z = 3. \end{cases}$$

(a) Express the intersection of these planes as a parametrized line. [Hint: Subtract the equations to obtain a new equation without x. Then let $t = z$ be a parameter and solve for x and y in terms of t.]

(b) We observe that $\mathbf{u} = \langle 1, 1, 2\rangle$ and $\mathbf{v} = \langle 1, -1, 1\rangle$ are normal vectors for the two planes. Compute the cross product $\mathbf{u} \times \mathbf{v}$. How is this vector related to the line in part (a)?

(a): We subtract (2) from (1) to obtain a new equation (3) that does not involve x:

$$(3) = (1) - (2) : 0 + 2y + z = -2.$$

Now subtract (3) from 2(1) to obtain a new equation (4) that does not involve y:

$$(4) = 2(1) - (3) : 2x + 0 + 3z = 4.$$

Thus we have solved for x and y in terms of z:

$$\begin{aligned} x &= 2 - (3/2)z \\ y &= -1 - (1/2)z. \end{aligned}$$

If we let $t = z$ be a parameter then we obtain a parametrized line:

$$\begin{cases} x &= 2 - (3/2)t, \\ y &= -1 - (1/2)t, \\ z &= t, \end{cases}$$

which can also be expressed as

$$\begin{aligned} \mathbf{r}(t) &= \langle x(t), y(t), z(t)\rangle \\ &= \langle 2 - (3/2)t, -1 - (1/2)t, t\rangle \\ &= \langle 2, -1, 0\rangle + t\langle -3/2, -1/2, 1\rangle. \end{aligned}$$

(b): On the other hand, let's consider the normal vectors of the planes (1) and (2), which are

$$\mathbf{u} = \langle 1, 1, 2\rangle \quad \text{and} \quad \mathbf{v} = \langle 1, -1, 1\rangle.$$

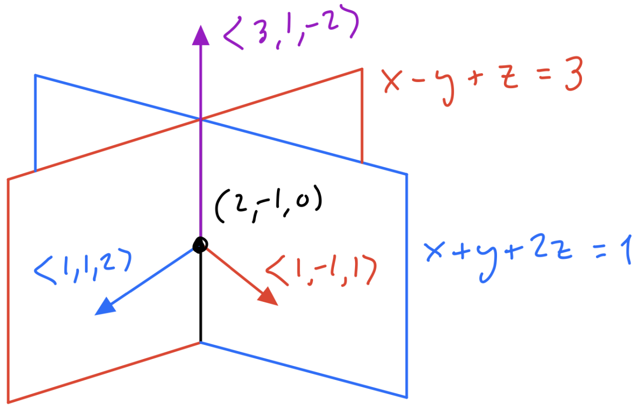
Their cross product is

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix} \\ &= \mathbf{i} \det \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} - \mathbf{j} \det \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} + \mathbf{k} \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \mathbf{i}(1 + 2) - \mathbf{j}(1 - 2) + \mathbf{k}(-1 - 1) \\ &= 3\mathbf{i} + 1\mathbf{j} - 2\mathbf{k} \\ &= \langle 3, 1, -2\rangle. \end{aligned}$$

We observe that $\mathbf{u} \times \mathbf{v}$ is a scalar multiple of the velocity vector from part (a):

$$\langle 3, 1, -2\rangle = -2\langle -3/2, -1/2, 1\rangle.$$

In fact, we could have used the cross product to solve (a) in a different way. Here is a picture (the red and blue vectors are perpendicular to the red and blue planes, respectively):



Problem 4. Projectile Motion. A projectile is launched from the point $(0,0)$ in $\mathbb{R}^2$ with an initial speed of s, at an angle of θ above the horizontal. Thus we have

$$\begin{aligned} \mathbf{r}(0) &= \langle 0, 0\rangle, \\ \mathbf{r}'(0) &= \langle s\cos\theta, s\sin\theta\rangle. \end{aligned}$$

Let $g > 0$ be the constant of gravity (which is $9.81\ m/s^2$ near the Earth).

(a) Use this information to compute the position $\mathbf{r}(t)$ at time t. [Hint: Neglecting air resistance, the acceleration due to gravity is constant: $\mathbf{r}''(t) = \langle 0, -g\rangle$.]

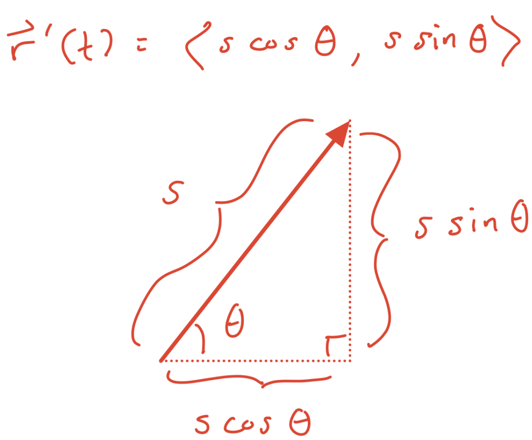
(b) Show that the particle travels a horizontal distance of $H = s^2 \sin(2\theta)/g$ before it hits the ground. [Hint: Use your answer $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ from part (a) and solve the equation $y(t) = 0$ for t. You will need the trig identity $\sin(2\theta) = 2\sin\theta\cos\theta$.]

(c) Find the value of θ that maximizes the horizontal distance traveled. [Hint: According to Calculus I, you should find the value of θ that makes $dH/d\theta = 0$. Recall that g and s are constant.]

(a): Fix some constants $s, \theta > 0$ and let the initial velocity be $\mathbf{r}'(0) = \langle s\cos\theta, s\sin\theta\rangle$. Then the initial speed is

$$\|\mathbf{r}'(0)\| = \sqrt{s^2\cos^2\theta + s^2\sin^2\theta} = \sqrt{s^2(\cos^2\theta + \sin^2\theta)} = \sqrt{s^2} = s.$$

That is, instead of specifying the initial velocity by its Cartesian coordinates, we will use the magnitude and direction. This idea is called "polar coordinates":



Our goal is to find explicit formulas for the position at time t. We begin by integrating $\mathbf{r}''(t) = \langle 0, -g\rangle$ to get $\mathbf{r}'(t)$. Since g is constant we have

$$\begin{aligned}\mathbf{r}'(t) &= \left\langle \int 0\,dt, \int -g\,dt \right\rangle \\ &= \langle c_1, -gt + c_2\rangle\end{aligned}$$

for come constants of integration c_1, c_2. We use the initial velocity to see that

$$\langle s\cos\theta, s\sin\theta\rangle = \mathbf{r}'(0) = \langle c_1, 0 + c_2\rangle = \langle c_1, c_2\rangle,$$

and hence

$$\mathbf{r}'(t) = \langle s\cos\theta, -gt + s\sin\theta\rangle.$$

Next we integrate $\mathbf{r}'(t)$ to get $\mathbf{r}(t)$. Since s, θ and g are constant we have

$$\begin{aligned}\mathbf{r}(t) &= \left\langle \int s\cos\theta\,dt, \int (-gt + s\cos\theta)\,dt \right\rangle \\ &= \left\langle (s\cos\theta)t + c_3, -\frac{1}{2}gt^2 + (s\sin\theta)t + c_4 \right\rangle\end{aligned}$$

for some constants c_3, c_4. We use the initial position to see that

$$\langle 0, 0\rangle = \mathbf{r}(0) = \langle 0 + c_3, 0 + 0 + c_4\rangle = \langle c_3, c_4\rangle,$$

and hence

$$\mathbf{r}(t) = \left\langle (s\cos\theta)t, -\frac{1}{2}gt^2 + (s\sin\theta)t \right\rangle.$$

(b): We want to know when the projectile hits the ground. In other words, we want to solve

$$\begin{aligned} y(t) &= 0 \\ -\frac{1}{2}gt^2 + (s\sin\theta)t &= 0 \\ t\left(-\frac{1}{2}gt + s\sin\theta\right) &= 0. \end{aligned}$$

We find that the projectile is on the ground at time $t = 0$ (of course) and also when

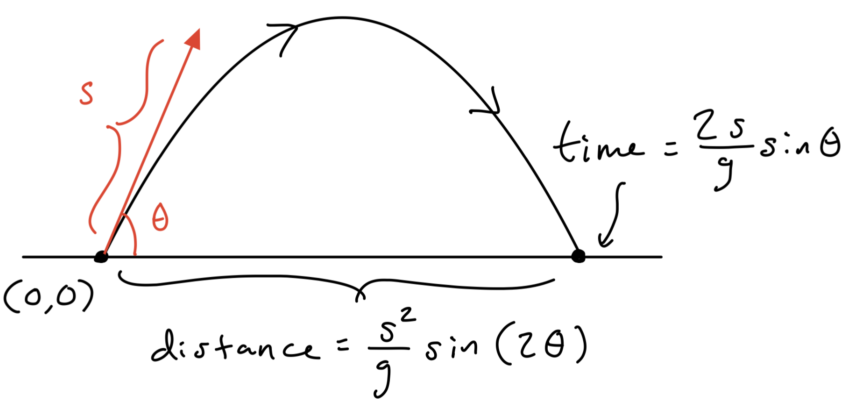
$$\begin{aligned} -\frac{1}{2}gt + s\sin\theta &= 0 \\ t &= \frac{2s}{g}\sin\theta. \end{aligned}$$

Now we want to know **where** the projectile hits the ground. Since it hits the ground at time $t = 2s\sin\theta/g$, the position when it hits the ground is[1]

$$\begin{aligned}\mathbf{r}\left(\frac{2s}{g}\sin\theta\right) &= \left\langle s\cos\theta\frac{2s}{g}\sin\theta, 0\right\rangle \\ &= \left\langle \frac{2s^2}{g}\sin\theta\cos\theta, 0\right\rangle \\ &= \left\langle \frac{s^2}{g}\sin(2\theta), 0\right\rangle.\end{aligned}$$

Here is a picture:

[1]Here I use the trig identity $\sin(2\theta) = 2\sin\theta\cos\theta$ to make the following computations simpler.



For which value of θ is the distance $s^2\sin(2\theta)/g$ maximized? To solve this we will think of the distance as a function of θ, with s and g fixed:

$$f(\theta) = \frac{s^2}{g}\sin(2\theta).$$

Then to maximize $f(\theta)$ we take the derivative with respect to θ and set this equal to zero:

$$\begin{aligned} df/d\theta &= 0 \\ \frac{s^2}{g}\cos(2\theta)\cdot 2 &= 0 \\ \cos(2\theta) &= 0. \end{aligned}$$

We conclude that $2\theta = 90°$ and hence $\theta = 45°$. Summary: The horizontal distance of a cannonball is maximized by aiming the cannon at $45°$ above the horizontal. This is true on any planet and for any initial speed.

Problem 5. Fun with the Product Rule. Recall the following "product rules" for vector-valued functions $\mathbf{f}, \mathbf{g} : \mathbb{R} \to \mathbb{R}^3$:

$$\begin{aligned} [\mathbf{f}(t) \bullet \mathbf{g}(t)]' &= \mathbf{f}'(t) \bullet \mathbf{g}(t) + \mathbf{f}(t) \bullet \mathbf{g}'(t), \\ [\mathbf{f}(t) \times \mathbf{g}(t)]' &= \mathbf{f}'(t) \times \mathbf{g}(t) + \mathbf{f}(t) \times \mathbf{g}'(t). \end{aligned}$$

(a) Let $\mathbf{r}(t)$ be the trajectory of a particle traveling on the surface of a sphere centered at $(0,0,0)$. In this case, show that $\mathbf{r}(t) \bullet \mathbf{r}'(t) = 0$ for all t. [Hint: By assumption we have $\|\mathbf{r}(t)\| = c$ for some constant c independent of t.]

(b) Let $\mathbf{r}(t)$ be the trajectory of a particle in space, and assume that $\mathbf{r}''(t) = c(t)\mathbf{r}(t)$ for some scalar function $c(t)$. In this case show that

$$[\mathbf{r}(t) \times \mathbf{r}'(t)]' = \langle 0,0,0 \rangle \quad \text{for all } t.$$

[Hint: Recall that $\mathbf{v} \times \mathbf{v} = \langle 0,0,0 \rangle$ for any vector $\mathbf{v}$.]

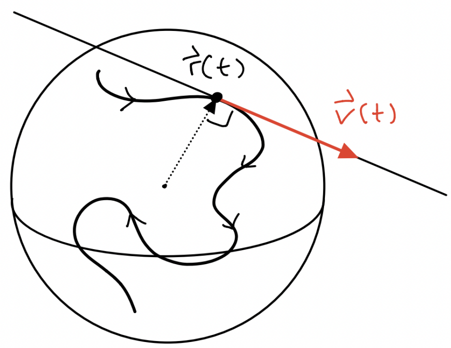
(a): If a particle travels on a sphere of radius c centered at $(0,0,0)$ then we must have $\|\mathbf{r}(t)\| = c$ for all t. Since $\|\mathbf{r}(t)\|^2 = \mathbf{r}(t) \bullet \mathbf{r}(t)$ we must also have

$$\begin{aligned} \|\mathbf{r}(t)\| &= c \\ \|\mathbf{r}(t)\|^2 &= c^2 \\ \mathbf{r}(t) \bullet \mathbf{r}(t) &= c^2. \end{aligned}$$

Now we take the derivative of both sides and apply the product rule:

$$
\begin{aligned}
[\mathbf{r}(t) \bullet \mathbf{r}(t)]' &= [c^2]' \\
\mathbf{r}'(t) \bullet \mathbf{r}(t) + \mathbf{r}(t) \bullet \mathbf{r}'(t) &= 0 && c^2 \text{ is constant} \\
\mathbf{r}(t) \bullet \mathbf{r}'(t) + \mathbf{r}(t) \bullet \mathbf{r}'(t) &= 0 \\
2\mathbf{r}(t) \bullet \mathbf{r}'(t) &= 0 \\
\mathbf{r}(t) \bullet \mathbf{r}'(t) &= 0.
\end{aligned}
$$

In other words, the velocity of the particle is always tangent to the sphere. Here is a picture:



(b): Let $\mathbf{r}(t)$ be the trajectory of a particle in $\mathbb{R}^3$ and assume that the acceleration and position vectors are in the same direction, i.e., that $\mathbf{r}''(t) = c(t)\mathbf{r}(t)$ for some scalar function $c(t)$. Then by using the product rule for the derivative of a cross product we find that

$$
\begin{aligned}
[\mathbf{r}(t) \times \mathbf{r}'(t)]' &= \mathbf{r}'(t) \times \mathbf{r}'(t) + \mathbf{r}(t) \times \mathbf{r}''(t) \\
&= \langle 0,0,0\rangle + \mathbf{r}(t) \times [c(t)\mathbf{r}(t)] \\
&= \langle 0,0,0\rangle + c(t)[\mathbf{r}(t) \times \mathbf{r}(t)] \\
&= \langle 0,0,0\rangle + c(t)\langle 0,0,0\rangle \\
&= \langle 0,0,0\rangle.
\end{aligned}
$$

This is a strange formula. We will explore its meaning in the next problem.

Problem 6. Universal Gravitation. Choose a coordinate system with the sun at the origin $(0,0,0)$ in $\mathbb{R}^3$. According to Newton, a planet at position $\mathbf{r}(t)$ feels a gravitational force $\mathbf{F}(t)$ pointed directly toward the sun, whose magnitude is

$$\|\mathbf{F}(t)\| = \frac{GMm}{\|\mathbf{r}(t)\|^2},$$

where M is the mass of the sun, m is the mass of the planet and G is a constant of gravitation. For simplicity, let's assume that $G = M = m = 1$.

(a) Show that $\mathbf{F}(t) = -GMm\mathbf{r}(t)/\|\mathbf{r}(t)\|^3$. It follows from Newton's Second Law that

$$\mathbf{r}''(t) = -\frac{GM}{\|\mathbf{r}(t)\|^3}\mathbf{r}(t).$$

[Hint: Since $\mathbf{F}(t)$ points directly toward the sun we must have $\mathbf{F}(t) = -c(t)\mathbf{r}(t)$ for some positive scalar $c(t)$, and hence $\|\mathbf{F}(t)\| = c(t)\|\mathbf{r}(t)\|$. Solve for $c(t)$.]

(b) **Conservation of Angular Momentum.** Consider the *angular momentum* vector

$$\mathbf{L}(t) = \mathbf{r}(t) \times \mathbf{r}'(t).$$

Use part (a) and Problem 5(b) to show that $\mathbf{L}'(t) = \langle 0, 0, 0 \rangle$ for all t. It follows that the angular momentum vector is constant.

(a): Since $\mathbf{F}(t) = -c(t)\mathbf{r}(t)$ for some positive scalar $c(t) > 0$ we have

$$\|\mathbf{F}(t)\| = \| - c(t)\mathbf{r}(t)\| = | - c(t)|\|\mathbf{r}(t)\| = c(t)\|\mathbf{r}(t)\|.$$

Then since $\|\mathbf{F}(t)\| = -GMm/\|\mathbf{r}(t)\|^2$ we have

$$\begin{aligned} \|\mathbf{F}(t)\| &= GMm/\|\mathbf{r}(t)\|^2 \\ c(t)\|\mathbf{r}(t)\| &= GMm/\|\mathbf{r}(t)\|^2 \\ c(t) &= GMm/\|\mathbf{r}(t)\|^3. \end{aligned}$$

Finally, Newton's second law gives

$$\begin{aligned} m\mathbf{r}''(t) &= \mathbf{F}(t) \\ m\mathbf{r}''(t) &= -c(t)\mathbf{r}(t) \\ m\mathbf{r}''(t) &= -\frac{GMm}{\|\mathbf{r}(t)\|^3}\mathbf{r}(t) \\ \mathbf{r}''(t) &= -\frac{GM}{\|\mathbf{r}(t)\|^3}\mathbf{r}(t). \end{aligned}$$

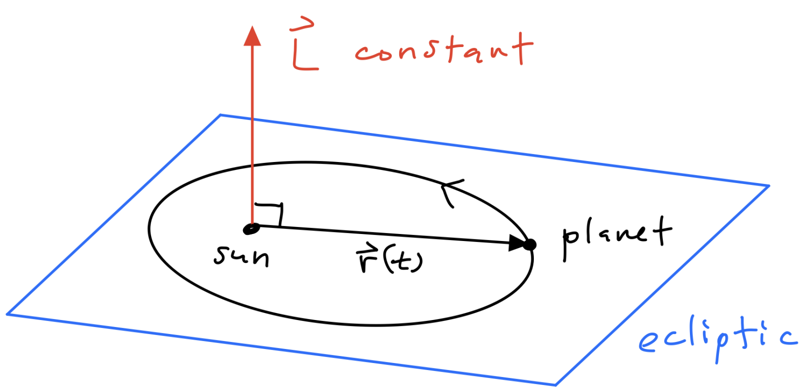
(b): Now we consider the angular momentum vector:[2]

$$\mathbf{L}(t) = \mathbf{r}(t) \times \mathbf{r}'(t).$$

From part (a) we know that $\mathbf{r}''(t) = c(t)\mathbf{r}(t)$ for some scalar $c(t)$, hence from Problem 5(b) we conclude that

$$\mathbf{L}'(t) = [\mathbf{r}(t) \times \mathbf{r}'(t)]' = \langle 0, 0, 0 \rangle.$$

In other words, the angular momentum vector $\mathbf{L}$ is constant. Since $\mathbf{L}$ is always perpendicular to $\mathbf{r}(t)$ and $\mathbf{r}'(t)$, this tells us, in particular, that the planet always stays in the plane that is perpendicular to $\mathbf{L}$, called the ecliptic. Here is a picture:



[2]Sorry, the true angular momentum is $m\mathbf{L}(t)$ where m is the mass of the planet.

For the Curious Only! (Everyone Else Please Ignore) If $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ then the vector differential equation

$$\mathbf{r}''(t) = -\frac{GM}{\|\mathbf{r}(t)\|^3}\mathbf{r}(t)$$

is equivalent to a system of three coupled nonlinear differential equations:

$$\begin{cases} x''(t) &= -GMx(t)/[x'(t)^2 + y'(t)^2 + z'(t)^2]^{3/2}, \\ y''(t) &= -GMy(t)/[x'(t)^2 + y'(t)^2 + z'(t)^2]^{3/2}, \\ z''(t) &= -GMz(t)/[x'(t)^2 + y'(t)^2 + z'(t)^2]^{3/2}. \end{cases}$$

One of Newton's great achievements was to show that these equations lead to the prediction of **elliptic planetary orbits**, which was earlier observed by Kepler without any explanation.

I will outline a modern proof of this using vector calculus. To simplify the formulas I will assume that $G = M = m = 1$.

- We showed in 6(b) that the angular velocity $\mathbf{L} = \mathbf{r}(t) \times \mathbf{r}'(t)$ is a constant vector.

- There is another conserved vector, called the *Runge-Lenz vector*:

  $$\mathbf{A}(t) = \mathbf{r}'(t) \times \mathbf{L} - \mathbf{r}(t)/\|\mathbf{r}(t)\|.$$

  One can check using identities for dot product and cross product that $\mathbf{A}'(t) = \langle 0, 0, 0\rangle$, hence $\mathbf{A}(t) = \mathbf{A}$ is constant. This is related to conservation of energy.

- Since $\mathbf{r}'(t) \times \mathbf{L}$ and $\mathbf{r}(t)/\|\mathbf{r}(t)\|$ are both perpendicular to $\mathbf{L}$, we see that $\mathbf{A}$ is perpendicular to $\mathbf{L}$. Thus we can choose a coordinate system so that $\mathbf{L} = \langle 0, 0, \ell\rangle$ and $\mathbf{A} = \langle e, 0, 0\rangle$ for some constants $e, \ell > 0$. The number e is some measure of energy.

- Since $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is perpendicular to $\mathbf{L} = \langle 0, 0, \ell\rangle$ we must have $z(t) = 0$ for all t. That is, the planet stays in the x, y-plane, which is called the "ecliptic".

- Our goal is to find formulas for $x(t)$ and $y(t)$. This is much easier if we switch to polar coordinates $r(t)$ and $\theta(t)$ where $x(t) = r(t)\cos[\theta(t)]$ and $y(t) = r(t)\sin[\theta(t)]$. Note in particular that that $r(t) = \sqrt{x(t)^2 + y(t)^2} = \|\mathbf{r}(t)\|$. To save space we will write r and θ instead of $r(t)$ and $\theta(t)$.

- By computing the expression $\mathbf{r}(t) \bullet (\mathbf{r}'(t) \times \mathbf{L})$ in two different ways (using various identities for dot product and cross product) one can show that

  $$r(1 + e\cos\theta) = \mathbf{r}(t) \bullet (\mathbf{r}'(t) \times \mathbf{L}) = \ell^2,$$

  and hence

  $$r = \ell^2/(1 + e\cos\theta).$$

  This is the equation of a "conic section" in polar coordinates.

- The constant e is called the "eccentricity" of the orbit. If $0 < e < 1$ then the orbit is an ellipse. If $e > 1$ then the planet has enough energy to escape the solar system and the orbit is a hyperbola.