Math 309 — Summer 2019
Quiz 3 — Drew Armstrong

1. Base b Arithmetic.

Convert the decimal number 111 into binary.

Set $q := 111$ and then repeatedly divide the quotient by 2:

$$\begin{aligned}
\mathbf{111} &= \mathbf{55}\cdot 2 && +1\\
\mathbf{55} &= \mathbf{27}\cdot 2 && +1\\
\mathbf{27} &= \mathbf{13}\cdot 2 && +1\\
\mathbf{13} &= \mathbf{6}\cdot 2 && +1\\
\mathbf{6} &= \mathbf{3}\cdot 2 && +0\\
\mathbf{3} &= \mathbf{1}\cdot 2 && +1\\
\mathbf{1} &= \mathbf{0}\cdot 2 && +1
\end{aligned}$$

We conclude that $111 = (1101111)_2$.

2. Induction Again. Fix some number $r \neq 1$.

Use induction to prove that $r^0 + r^1 + \cdots + r^n = (r^{n+1}-1)/(r-1)$ for all $n \geq 0$.

Proof. For the base case $n = 0$ we observe that

$$r^0 = \frac{r^1 - 1}{r-1} \quad \text{is a true statement.}$$

Now fix some integer $n \geq 0$ and assume for induction that

$$r^0 + r^1 + \cdots + r^n = \frac{r^{n+1}-1}{r-1} \quad \text{is a true statement.}$$

Then we also have

$$\begin{aligned}
r^0 + r^1 + \cdots + r^{n+1} &= r^0 + r^1 + \cdots + r^n + r^{n+1}\\
&= \frac{r^{n+1}-1}{r-1} + r^{n+1}\\
&= \frac{r^{n+1}-1}{r-1} + \frac{r^{n+1}(r-1)}{r-1}\\
&= \frac{r^{n+1}-1}{r-1} + \frac{r^{n+2}-r^{n+1}}{r-1}\\
&= \frac{r^{n+1}-1+r^{n+2}-r^{n+1}}{r-1}\\
&= \frac{r^{n+2}-1}{r-1}.
\end{aligned}$$

Hence the statement is true for $n+1$. $\square$

2. Division With Remainder.

(a) Accurately State the Division Theorem.

For all integers $a, b \in \mathbb{Z}$ with $b > 0$, there exist unique integers $q, r \in \mathbb{Z}$ such that

$$\begin{cases} a = qb + r, \\ 0 \leq r < b. \end{cases}$$

(b) Use the Euclidean algorithm to compute $\gcd(100, 23)$.

First we divide 100 by 23 to get sone remainder r. Then we replace the pair $(100, 23)$ by $(23, r)$ and repeat:

$$\begin{aligned} \mathbf{100} &= 4 \cdot \mathbf{23} && +\mathbf{8} \\ \mathbf{23} &= 2 \cdot \mathbf{8} && +\mathbf{7} \\ \mathbf{8} &= 1 \cdot \mathbf{7} && +\mathbf{1} \\ \mathbf{7} &= 7 \cdot \mathbf{1} && +\mathbf{0} \end{aligned}$$

We conclude that $\gcd(100, 23) = 1$.

(c) Apply your work from (b) to find the continued fraction expansion of $100/23$.

The sequence of quotients $(4, 2, 1, 7)$ from part (b) tells us that

$$\frac{100}{23} = 4 + \frac{1}{2 + \frac{1}{1 + \frac{1}{7}}}.$$