Math 309 **Summer 2019**
Homework 5 **Drew Armstrong**

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1. Check that the relation $\sum_{u \in V_G} \deg(u) = 2 \cdot \#E_G$ holds for each of the following:

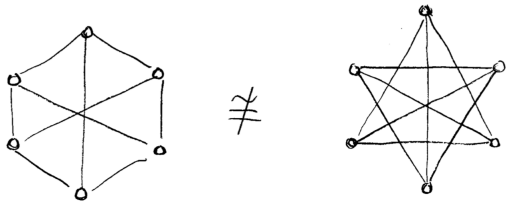
(a) Cycles C_n
(b) Paths P_n
(c) Complete graphs K_n
(d) Complete bipartite graphs $K_{m,n}$

2. The *hypercube graph* Q_n has 2^n vertices corresponding to the binary strings of length n and edges corresponding to "flipping one bit."

(a) Draw the graphs Q_1, Q_2, Q_3.
(b) Compute the number of edges in Q_n. [Hint: What are the vertex degrees?]

3. Explain why every graph has two vertices of the same degree. [Hint: Suppose that the graph has n vertices. Show that the degrees 0 and $n-1$ cannot both occur. So how many possible degree values are there?]

4. Give two different proofs that the following graphs are not isomorphic:



(a) Show that the complements are not isomorphic.
(b) Show that the left graph is bipartite, while the right graph is not.

5. Let G be a graph with n vertices.

(a) If $2 \leq k \leq n$ show that $\binom{n-k+1}{2} \leq \binom{n-1}{2}$.
(b) If G has more than $\binom{n-1}{2}$ edges, prove that G is connected. [Hint: Let k be the number of connected components of G. There is a relevant theorem in the notes.]
(c) Draw a graph with 6 vertices and $\binom{5}{2}$ edges that is **not** connected.

6. Let $G = (V, E)$ be a bipartite graph with partition $V = A \cup B$. In other words, assume that every edge of the graph has the form $\{a, b\}$ for some $a \in A$ and $b \in B$.

(a) Let $\deg(A), \deg(B)$ be the average degree of a vertex in A, B, respectively. Prove that

$$\#A \cdot \deg(A) = \#B \cdot \deg(B).$$

(b) A certain statistical survey[1] found that men in the United States have 74% more opposite sex partners than women. Explain why this result cannot possibly be accurate. (Just the math, please.) [Hint: Let A and B be the sets of men and women.]

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[1] *The Social Organization of Sexuality* (1994) by Edward O. Laumann et al. The authors themselves acknowledge (pg. 185) that this result cannot be accurate.