Math 309 **Summer 2019**
Homework 3 **Drew Armstrong**

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1. Let $a, b \in \mathbb{N}$. Use the definition of integers from the notes to prove that

$$ab = 0 \qquad \Longrightarrow \qquad (a = 0) \vee (b = 0).$$

2. Here is a false proof. Find the mistake.

Claim. The following statement is true for all $n \in \mathbb{N}$:

$$P(n) = \text{“if } a, b \in \mathbb{N} \text{ satisfy } n = \max(a, b) \text{ then } a = b.\text{”}$$

Proof. Clearly $P(0)$ is true because $a, b \in \mathbb{N}$ and $\max(a, b) = 0$ imply $a = b = 0$. Now fix some $n \geq 0$ and assume for induction that $P(n)$ is true. In order to prove that $P(n + 1)$ is also true we consider any numbers $a, b \in \mathbb{N}$ with $\max(a, b) = n + 1$. But then we have $\max(a - 1, b - 1) = n$ and $P(n)$ implies that $a - 1 = b - 1$, hence $a = b$. $\square$

3. Given $a, b \in \mathbb{N}$ we define the following notation:

$$\text{“}a|b\text{”} = \text{“}a \text{ divides } b\text{”} = \text{“}\exists k \in \mathbb{N}, ak = b.\text{”}$$

We say that $n \in \mathbb{N}$ is *not prime* if there exist $a, b \in \mathbb{N}$ with $n = ab$ and $a, b \in \{2, 3, \ldots, n - 1\}$. (We say that a and b are *proper factors* of n.) Now consider the following statement:

Every natural number $n \geq 2$ is divisible by a prime number.

(a) Prove the statement by strong induction.
(b) Prove the statement by well-ordering.

3. Convert the decimal number 123456789 into the following base systems:

(a) Binary $\{0, 1\}$
(b) Ternary $\{0, 1, 2\}$
(c) Hexadecimal $\{0, 1, \ldots, 9, A, B, \ldots, F\}$

5. Convert the decimal numbers 12 and 23 into binary. Multiply them in binary. Then convert the result back into decimal notation.

6. Euclidean Algorithm.

(a) Apply the Euclidean Algorithm to compute the gcd of 3094 and 2513.
(b) Repeat the same sequence of steps to find the continued fraction expansion of 3094/2513:

$$\frac{3094}{2513} = q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cfrac{1}{q_4 + \cdots}}}.$$

7. $\sqrt{2}$ **is Irrational.** If a and b are integers then the Euclidean Algorithm guarantees that the continued fraction expansion of a/b is **finite**. Prove that

$$\sqrt{2} = 1 + \frac{1}{1 + \sqrt{2}}$$

and use this to show that the continued fraction expansion of $\sqrt{2}$ is **infinite**. It follows that $\sqrt{2}$ is not a fraction of integers.