Math 309 **Summer 2014**
Final Exam **Drew Armstrong**

1. Boolean Algebra.

(a) Draw the truth table for $P \Rightarrow Q$.

Here is the truth table. For fun, we also observe that $P \Rightarrow Q = (\neg P) \vee Q$.

P	Q	$\neg P$	$(\neg P) \vee Q$	$P \Rightarrow Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

(b) Prove that $(P \Rightarrow Q) = (\neg Q \Rightarrow \neg P)$ using a truth table.

Observe that the final columns in both tables are the same.

P	Q	$\neg Q$	$\neg P$	$\neg Q \Rightarrow \neg P$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

Alternatively, here's an abstract-algebraic proof:

$$(\neg Q \Rightarrow \neg P) = ((\neg\neg Q) \vee \neg P) = (Q \vee \neg P) = (\neg P \vee Q) = (P \Rightarrow Q).$$

(c) Express the statement $P \Leftrightarrow Q$ using only the boolean operations $\vee, \wedge, \neg$.

Recall that $P \Leftrightarrow Q$ means $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. Therefore from the observation in part (a) we can write

$$(P \Leftrightarrow Q) = (P \Rightarrow Q) \wedge (Q \Rightarrow P) = (\neg P \vee Q) \wedge (\neg Q \vee P).$$

Alternatively, we could first draw the truth table:

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Since this function has T's in the $P \wedge Q$ row and the $\neg P \wedge \neg Q$ row, the disjunctive normal form is

$$(P \Leftrightarrow Q) = (P \wedge Q) \vee (\neg P \wedge \neg Q).$$

2. Induction. Your goal in this problem is to prove the following identity by induction.

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1.$$

(a) State **exactly** what you want to prove. Make sure to define $P(n)$.

For all integers $n \geq 1$ we define the statement

$$P(n) = \text{“}1 \cdot 1! + 2 \cdot 2! + \cdots n \cdot n! = (n+1)! - 1.\text{”}$$

We will use induction to prove that $P(n)$ is true for all $n \geq 1$.

(b) State and prove the base case.

We observe that the statement $P(1)$ is true:

$$P(1) = \text{“}1 \cdot 1! = (1+1)! - 1\text{”} = \text{“}1 = 2 - 1\text{”} = T.$$

(c) State the prove the induction step.

Now consider an arbitrary integer $k \geq 1$ and let us assume for induction that $P(k)$ is true. In other words, let us assume that

$$1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k+1)! - 1.$$

But then we have

$$\begin{aligned}
&1 \cdot 1! + 2 \cdot 2! + \cdots + (k+1) \cdot (k+1)! \\
&= [1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k!] + (k+1) \cdot (k+1)! \\
&= [(k+1)! - 1] + +(k+1) \cdot (k+1)! && \text{induction} \\
&= [(k+1)! + (k+1) \cdot (k+1)!] - 1 \\
&= [1 + (k+1)] \cdot (k+1)! - 1 \\
&= (k+2) \cdot (k+1)! - 1 \\
&= (k+2)! - 1,
\end{aligned}$$

which means that $P(k+1)$ is also true.

[Remark: Where did I come up with this identity? Consider the collection of all words that can be made with the symbols $a_1, a_2, \ldots, a_{n+1}$. We will say the the symbol a_i is “happy” if it is placed in the ith position from the left. Note that every word except $a_1 a_2 \cdots a_{n+1}$ has at least one unhappy symbol. Therefore the number of words with at least one unhappy symbol is $(n+1)! - 1$. On the other hand, let us consider the collection of words in which the **leftmost unhappy symbol** occurs in the kth position from the **right**. One can argue that there are $(k-1) \cdot (k-1)!$ such words. Now sum over k.]

3. Binomial Theorem.

(a) Accurately state the Binomial Theorem.

Fix a non-negative integer $n \geq 0$. Then for all numbers x and y we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k y^{n-k}.$$

(b) Prove that a set with n elements has an equal number of "even subsets" (subsets with an even number of elements) and "odd subsets" (subsets with an odd number of elements). [Hint: Just plug something in.]

Since the binomial theorem is true for all numbers x and y, we may substitute $x = -1$ and $y = 1$ to obtain

$$\begin{aligned}(-1+1)^n &= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + \binom{n}{n}(-1)^n \\ 0 &= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + \binom{n}{n}(-1)^n \\ \binom{n}{1} + \binom{n}{3} + \cdots &= \binom{n}{0} + \binom{n}{2} + \cdots\end{aligned}$$

Since $\binom{n}{k}$ is the number of subsets with size k, the last equation tells us that the number of odd-sized subsets equals the number of even-sized subsets.

(c) How many subsets of $\{1,2,3,4,5,6\}$ have an **even** number of elements?

The total number of subsets of $\{1,2,3,4,5,6\}$ is

$$2^{\#\{1,2,3,4,5,6\}} = 2^6 = 64.$$

Now let E and O be the numbers of even and odd subsets, so that $E + O = 64$. But we know from part (b) that $E = O$, so that

$$\begin{aligned}E + O &= 64 \\ E + E &= 64 \\ 2E &= 64 \\ E &= 32.\end{aligned}$$

[Remark: In general, the number of even subsets of $\{1, 2 \ldots, n\}$ is 2^{n-1}.]

4. Probability. Consider a biased coin with $P(\text{"heads"}) = 1/3$.

(a) If you flip the coin n times. What is the probability that you get "heads" **exactly** k times?

The probability of getting heads exactly k times in n flips of a coin is

$$\binom{n}{k} P(\text{"heads"})^k P(\text{"tails"})^{n-k} = \binom{n}{k}\left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{n-k} = \binom{n}{k}\frac{2^{n-k}}{3^n}$$

(b) If you flip the coin 5 times, what is the probability that you get "heads" an **even** number of times?

In this case we have $n = 5$. To compute the probability of an even number of heads, we sum the probabilities from (a) over all even values of k:

$$\begin{aligned}\binom{5}{0}\frac{2^{5-0}}{3^5} + \binom{5}{2}\frac{2^{5-2}}{3^5} + \binom{5}{4}\frac{2^{5-4}}{3^5} &= \binom{5}{0}\frac{32}{243} + \binom{5}{2}\frac{8}{243} + \binom{5}{4}\frac{2}{243} \\ &= 1 \cdot \frac{32}{243} + 10 \cdot \frac{8}{243} + 5 \cdot \frac{2}{243} \\ &= \frac{122}{243} = 50.2\%\end{aligned}$$

(c) If you flip the coin 111 times, how many times do you **expect** to get "heads"?

Consider a general coin with $P(\text{"heads"}) = p$ and $P(\text{"tails"}) = 1 - p$. If we flip this coin n times then on average we will expect to get heads pn times.

Since our coin has $p = 1/3$, if we flip the coin $n = 111$ times then on average we expect to see heads

$$np = 111 \cdot 1/3 = 37 \text{ times.}$$

5. Integers.

(a) Accurately state the Division Theorem for integers. [Hint: For all $a, b \in \mathbb{Z}$ with $b \neq 0$ ...]

For all integers $a, b \in \mathbb{Z}$ with $b \neq 0$, there exist unique integers $q, r \in \mathbb{Z}$ satisfying the following two properties:

$$\begin{cases} a = qb + r, \\ 0 \leq r < |b|. \end{cases}$$

(b) Accurately state the definition of an "even" integer.

We say an integer is even if it is "divisible by 2." In other words:

$$\text{"}n \text{ is even"} = \text{"}2|n\text{"} = \text{"}\exists k \in \mathbb{Z}, 2k = n.\text{"}$$

(c) Consider an integer $n \in \mathbb{Z}$. Prove that if n^2 is even then n is even.

We wish to prove that $2|n^2$ implies $2|n$. In order to do this we will instead prove the (equivalent) contrapositive statement that $2 \nmid n$ implies $2 \nmid n^2$. We will also use the fact (proved from the division theorem) that every non-even (i.e., odd) number has the form $2k + 1$ for some $k \in \mathbb{Z}$.

So let us suppose that $n \in \mathbb{Z}$ is odd; say $n = 2k + 1$ for some $k \in \mathbb{Z}$. It follows that

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2 \cdot (\text{some integer}) + 1$$

is also odd. □

6. Bonus. Give a **counting proof** of the following identity:

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Proof: Consider integers $0 \leq k \leq n$. From a bag of n unlabeled apples we will choose k apples to receive stickers. One of these k apples will receive **two** stickers and the other $k-1$ will receive **one** sticker each. We will count the possibilities in two ways.

On the one hand, we can choose the k stickered apples in $\binom{n}{k}$ ways. Then there are $k = \binom{k}{1}$ ways to choose the apple that will receive two stickers. This gives a total of

$$\binom{n}{k} \times k \quad \text{choices.}$$

On the other hand, we could first choose the two-stickered apple. There are $n = \binom{n}{1}$ ways to do this. Then we could choose $k-1$ apples from the remaining $n-1$ apples to receive one sticker each. There are $\binom{n-1}{k-1}$ ways to do this, for a total of

$$n \times \binom{n-1}{k-1} \quad \text{choices.}$$

Since these two formulas count the same things, they must be equal.

Math 309 **Summer B 2018**
Final Exam **Drew Armstrong**

1. Boolean Algebra.

(a) Use a truth table to prove that $\neg(P \vee Q) = (\neg P \wedge \neg Q)$ for all $P, Q \in \{T, F\}$.

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

(b) Recall that we define $(P \Rightarrow Q) := (\neg P) \vee Q$ for all $P, Q \in \{T, F\}$. Use this definition to prove that $(P \Rightarrow Q) = (\neg Q \Rightarrow \neg P)$. [No truth table is necessary.]

$$(\neg Q \Rightarrow \neg P) = (\neg(\neg Q) \vee \neg P) = (Q \vee \neg P) = (\neg P \vee Q) = (P \Rightarrow Q)$$

(c) Use parts (a) and (b) to prove that $(P \Rightarrow (Q \vee R)) = ((\neg Q \wedge \neg R) \Rightarrow \neg P)$ for all $P, Q, R \in \{T, F\}$. [No truth table is necessary.]

$$\begin{aligned} ((\neg Q \wedge \neg R) \Rightarrow \neg P) &= (\neg(\neg Q \wedge \neg R) \vee \neg P) && \text{(b)} \\ &= ((\neg\neg Q \vee \neg\neg R) \vee \neg P) && \text{(a)} \\ &= ((Q \vee R) \vee \neg P) \\ &= (\neg P \vee (Q \vee R)) \\ &= (P \Rightarrow (Q \vee R)) && \text{(b)} \end{aligned}$$

2. Even and Odd. The division theorem tells us that every integer $n \in \mathbb{Z}$ has one of the following two forms, which we call *even* and *odd*:

$$n = \begin{cases} 2k + 0 & \text{for some } k \in \mathbb{Z}, \\ 2k + 1 & \text{for some } k \in \mathbb{Z}. \end{cases}$$

(a) Prove that for all $n \in \mathbb{Z}$ we have (n is even) $\Rightarrow$ (n^2 is even).

Proof. Assume that $n \in \mathbb{Z}$ is even so that $n = 2k$ for some $k \in \mathbb{Z}$. Then we have

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2),$$

and hence n^2 is also even. □

(b) Prove that for all $n \in \mathbb{Z}$ we have (n^2 is even) $\Rightarrow$ (n is even).

Proof. We sill prove the contrapositive statement: (n is odd) $\Rightarrow$ (n^2 is odd). So assume that $n \in \mathbb{Z}$ satisfies $n = 2k+1$ for some $k \in \mathbb{Z}$. Then we have

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1,$$

and hence n^2 is also odd. □

(c) Prove for all $m, n \in \mathbb{Z}$ that (mn is even) $\Rightarrow$ (m is even or n is even). [Hint: See Problem 1.]

Proof. From Problem 1(c) this statement is equivalent to the following:

$$(m \text{ and } n \text{ are both odd}) \Rightarrow (mn \text{ is odd}).$$

So assume that $m, n \in \mathbb{Z}$ are both odd. By definition this means that $m = 2k+1$ and $n = 2\ell + 1$ for some integers $k, \ell \in \mathbb{Z}$. But then we have

$$mn = (2k+1)(2\ell+1) = 4k\ell + 2k + 2\ell + 1 = 2(2k\ell + k + \ell) + 1,$$

which implies that mn is odd. □

3. Induction. We will prove by induction that the following statement is true for all $n \geq 1$:

$$P(n) = \text{“}2 + 4 + 6 + \cdots + 2n = n(n+1).\text{”}$$

(a) Accurately state (some version of) the Principle of Induction.

If ($P(1) = T$) and if ($\forall n \geq 1, P(n) \Rightarrow P(n+1)$) then ($\forall n \geq 1, P(n) = T$).

(b) Verify the base case(s).

$$P(1) = \text{“}2 = 1(1+1)\text{”} = T.$$

(c) Prove the induction step.

Proof. Fix some $n \geq 1$ and assume for induction that $P(n)$ is true, i.e., that

$$2 + 4 + 6 + \cdots + 2n = n(n+1).$$

Then $P(n+1)$ is also true because

$$\begin{aligned} 2 + 4 + 6 + \cdots + 2(n+1) &= (2 + 4 + 6 + \cdots + 2n) + 2(n+1) \\ &= n(n+1) + 2(n+1) \\ &= (n+1)(n+2). \end{aligned}$$

□

4. Counting.

(a) Let S be a set with 10 elements. In how many ways can we choose a subset of size 3?

$$\binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$$

(b) How many words can be formed using all six of the letters b, a, n, a, n, a ?

$$\binom{6}{1,3,2} = \frac{6!}{1!3!2!} = \frac{6 \cdot 5 \cdot 4}{2} = 60$$

(c) Suppose that a deck contains 3 red cards and 4 blue cards, and suppose that 3 (unordered) cards are drawn at random. Compute the ratio A/B, where

$A =$ the number of ways to get 1 red and 2 blue cards,
$B =$ the number of possibilities when color doesn't matter.

To compute A we choose 1 red card from 3 and then choose 2 blue cards from 4:

$$A = \underbrace{\binom{3}{1}}_{\text{choose 1 red}} \times \underbrace{\binom{4}{2}}_{\text{choose 2 blue}} = \binom{3}{1}\binom{4}{2} = 3 \cdot 6 = 18.$$

To compute B we simply choose 3 cards from 7:

$$B = \binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35.$$

Thus the ratio is $A/B = 18/35 = 51.4\%$. [Remark: This is the probability of getting 1 red and 2 blue cards when 3 cards are drawn at random.]

5. Induction Without Algebra. Consider the following statement:

$$P(n) = \text{"The set } \{1, 2, \ldots, n\} \text{ has } 2^n \text{ different subsets."}$$

(a) Prove that $P(3)$ is a true statement by writing down all of the subsets.

There are $2^3 = 8$ subsets of $\{1, 2, 3\}$:

	$\{1,2,3\}$	
$\{1,2\}$	$\{1,3\}$	$\{2,3\}$
$\{1\}$	$\{2\}$	$\{3\}$
	$\emptyset$	

(b) If $n \geq 2$, describe an explicit bijection between the following two sets:

$$\left\{\text{all subsets of } \{1,2\ldots,n-1\}\right\} \leftrightarrow \left\{\text{subsets of } \{1,2,\ldots,n\} \text{ that } \textbf{must} \text{ contain } n\right\}$$

Given a subset $A \subseteq \{1,2,\ldots,n-1\}$ we add n to get $B := A \cup \{n\} \subseteq \{1,2,\ldots,n\}$. Conversely, given a subset $B \in \{1,2,\ldots,n\}$ we subtract n to get $A := B \cap \{n\}' \subseteq \{1,2,\ldots,n-1\}$. Note that these two operations are inverses.

(c) By dividing the subsets of $\{1,2,\ldots,n\}$ into those that either do or do not contain the symbol n, give an inductive proof that $P(n)$ is true for all $n \geq 1$.

(Bonus points may be awarded for style.)

Proof. I will prove by induction that for all $n \geq 1$ the set $\{1,2,\ldots,n\}$ has 2^n subsets. When $n = 1$ we observe that the statement is true because the set $\{1\}$ has exactly $2 = 2^1$ subsets: $\{1\}$ and $\emptyset$. Now fix some $n \geq 1$ and assume for induction that $\{1,2,\ldots,n-1\}$ has 2^{n-1} subsets. In this case we will prove that $\{1,2,\ldots,n\}$ has 2^n subsets.

So let S be the set of subsets of $\{1,2,\ldots,n\}$ and break this into two pieces as follows:

$$S' := \{A \subseteq \{1,2,\ldots,n\} : n \notin A\},$$
$$S'' := \{A \subseteq \{1,2,\ldots,n\} : n \in A\}.$$

We observe that S' is the set of all subsets of $\{1,2,\ldots,n-1\}$ hence by induction we know that $\#S' = 2^{n-1}$. Furthermore, from part (b) we have a bijection $S' \leftrightarrow S''$ so that $\#S'' = \#S' = 2^{n-1}$. Finally, we conclude that

$$\#S = \#S' + \#S'' = 2^{n-1} + 2^{n-1} = 2 \cdot 2^{n-1} = 2^n$$

as desired. $\square$