Math 309 **Summer B 2018**
Quiz 5 **Drew Armstrong**

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1. Accurately state (some version of) the Principle of Induction.

For each integer $n \geq 1$ let $P(n)$ be a logical statement. Suppose that the following two conditions hold:

- $P(1)$ is a true statement, and
- For all integers $k \geq 1$ we have $P(k) \Rightarrow P(k+1)$.

Then we may conclude that $P(n)$ is a true statement for all $n \geq 1$.

2. Consider the sequence of numbers defined recursively by

$$\begin{cases} r_0 = 1 & \text{and} \\ r_n = 2 \cdot r_{n-1} & \text{for all } n \geq 1. \end{cases}$$

Now we will prove that the the following statement holds for all integers $n \geq 0$:

$$P(n) = \text{“}r_n = 2^n.\text{”}$$

(a) Verify the base case(s).

We observe that $P(0) =$ “$r_0 = 2^0 = 1$” is a true statement.

(b) Prove the induction step.

Consider any integer $k \geq 0$ and let us assume for induction that $P(k)$ is a true statement. That is, let us assume that $r_k = 2^k$. But then we have

$$\begin{aligned} r_{k+1} &= 2 \cdot r^k && \text{definition} \\ &= 2 \cdot 2^k && \text{assumption} \\ &= 2^{k+1}, \end{aligned}$$

which proves that the statement $P(k+1) =$ “$r_{k+1} = 2^{k+1}$” is also true.

3. Consider the sequence of numbers defined recursively by

$$\begin{cases} s_1 = 1 & \text{and} \\ s_2 = 3 & \text{and} \\ s_n = s_{n-1} + s_{n-2} & \text{for all } n \geq 3. \end{cases}$$

(a) Complete the following table:

n	1	2	3	4	5	6
s_n	1	3	4	7	11	18

(b) Now we will prove by induction that the following statement holds for all $n \geq 1$:

$$P(n) = \text{“}s_n < 2^n.\text{”}$$

- Verify the base case(s).

  We observe that the first two statements are true:

$$P(1) = \text{“}s_1 < 2\text{”} = \text{“}1 < 2,\text{”}$$
$$P(2) = \text{“}s_2 < 4\text{”} = \text{“}3 < 4.\text{”}$$

- Prove the induction step.

  Now consider any $k \geq 3$ and let us assume for induction that the statements $P(k-1)$ and $P(k-2)$ are true. That is, let us assume that $s_{k-1} < 2^{k-1}$ and $s_{k-2} < 2^{k-2}$. But then we have

$$\begin{aligned} s_k &= s_{k-1} + s_{k-2} && \text{definition} \\ &< 2^{k-1} + 2^{k-2} && \text{induction} \\ &< 2^{k-1} + 2^{k-1} && \text{since } 2^{k-2} < 2^{k-1} \\ &= 2 \cdot 2^{k-1} \\ &= 2^k, \end{aligned}$$

  which proves that the statement $P(k) = \text{“}s_k < 2^k\text{”}$ is also true.