Math 309 — Summer B 2018
Homework 5 — Drew Armstrong

1. De Morgan's Law. For all integers $n \geq 1$ let $P(n)$ be the following statement:

"For any n statements $Q_1, Q_2, \ldots Q_n \in \{T, F\}$ we have $\neg(Q_1 \wedge \cdots \wedge Q_n) = \neg Q_1 \vee \cdots \vee \neg Q_n$."

Use induction to prove that $P(n)$ is true for all $n \geq 1$. [Hint: You proved on HW2 that $P(2)$ is a true statement. You do not need to prove this again.]

2. Euclid's Lemma. Let $p \in \mathbb{Z}$ be prime.

(a) For all integers $a, b \in \mathbb{Z}$ prove that

$$(p|ab) \Rightarrow (p|a \vee p|b).$$

[Hint: It is equivalent to prove $(p|ab \wedge p \nmid a) \Rightarrow p|b$. Use HW3.]

(b) For all integers $n \geq 1$ we define the statement $P(n)$ as follows:

"For any n integers $a_1, a_2, \ldots, a_n \in \mathbb{Z}$ we have $(p|a_1 a_2 \cdots a_n) \Rightarrow (p|a_i$ for some $i)$."

Use induction to prove that $P(n)$ is true for all $n \geq 1$. [Hint: Part (a) is $P(2)$.]

3. Multiplicative Cancellation. For all integers $n \geq 1$ let $P(n)$ be the following statement:

"$\forall m \geq 1, mn \geq 1$."

(a) Show that $P(1)$ is a true statement.

(b) Consider any integer $k \geq 1$ and assume for induction that $P(k)$ is a true statement. In this case, prove that $P(k+1)$ is also a true statement.

(c) Use the result of (a) and (b) to prove the following:

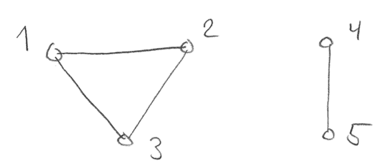
$$\forall a, b \in \mathbb{Z}, (ab = 0) \Rightarrow (a = 0 \vee b = 0).$$

[Hint: It is equivalent to prove $(a \neq 0 \wedge b \neq 0) \Rightarrow (ab \neq 0)$. If $a \neq 0$ and $b \neq 0$ then we must have $m = |a| \geq 1$ and $n = |b| \geq 1$.]

(d) Use the result of part (c) to prove the following:

$$\forall a, b, c \in \mathbb{Z}, (ab = ac \wedge a \neq 0) \Rightarrow (b = c).$$

4. A Graph Theory Problem. A *simple graph* consists of a set V of *vertices*, together with a set E of unordered pairs of vertices, called *edges*. For example, the following graph has $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{4, 5\}\}$:



We say that a graph is *connected* if for all pairs of vertices $u, v \in V$ there exists some sequence of edges $\{u_1, u_2\}, \{u_2, u_3\}, \ldots, \{u_\ell, u_{\ell+1}\}$ starting with $u_1 = u$ and ending with $u_{\ell+1} = v$. (The graph in the example is **not** connected.)

Use induction to prove that every connected graph with n vertices has at least $n - 1$ edges.