Math 309 Summer B 2018
Homework 3 Drew Armstrong

1. Poker Hands. A standard deck contains 52 cards. Half the cards are red and half are black. A "poker hand" consists of 5 cards chosen at random from the deck.

(a) How many different poker hands are there?
(b) How many poker hands contain all red cards?
(c) How many poker hands contain 1 red and 4 black cards? [Hint: Choose the red card first, then choose the black cards.]

(a) The cards are unordered and may not be repeated. So the number of poker hands is

$$\binom{52}{5} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2,598,960.$$

(b) The number of ways to choose 5 red cards is

$$\binom{26}{5} = \frac{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 65,780.$$

(c) There are 26 ways to choose a red card and there are $\binom{26}{4}$ ways to choose 4 black cards. The total number of choices is

$$26 \times \binom{26}{4} = 26 \times \frac{26 \cdot 25 \cdot 24 \cdot 23}{4 \cdot 3 \cdot 2 \cdot 1} = 388,700.$$

Remark: More generally, the number of poker hands with k red and $5-k$ black cards is

$$\binom{26}{k} \times \binom{26}{5-k}.$$

By summing over all possible values of k we obtain an interesting identity:

$$(\text{total \# poker hands}) = \sum_{k=0}^{5} (\text{\# hands with } k \text{ red cards})$$
$$\binom{52}{5} = \sum_{k=0}^{5} \binom{26}{k}\binom{26}{5-k}.$$

Even more generally, suppose there are n cards in the deck. Suppose that r cards are red and b cards are black, so that $r+b=n$, and suppose that a poker hand consists of h cards drawn at random. Then our identity becomes

$$\binom{n}{h} = \sum_{k=0}^{h} \binom{r}{k}\binom{b}{h-k}.$$

2. Double Counting. In this problem you will give two proofs of the identity

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

(a) Prove the identity using pure algebra. [Hint: $n! = n \times (n-1)!$]

(b) In a certain classroom of n students we want to choose a committee of k students, one of which will be the president of the committee. Prove the identity by counting the possible choices in two different ways. [Hint: Will you choose the president before or after choosing the committee members?]

(a) Pure algebra:

$$k\binom{n}{k} = \frac{k}{k!} \cdot \frac{n!}{(n-k)!} = \frac{1}{(k-1)!} \cdot \frac{n \cdot (n-1)!}{(n-k)!} = n \cdot \frac{(n-1)!}{(k-1)!(n-k)!} = n\binom{n-1}{k-1}.$$

(b) Counting argument: Suppose we want to choose a committee of k students from a classroom of n students. One committee member will be the president. On the one hand, there are $\binom{n}{k}$ ways to choose the committee and then k ways to choose the president, for a total of

$$k\binom{n}{k} \text{ choices.}$$

On the other hand, suppose we choose the president first. There are n ways to do this. Then we must choose the remaining $k-1$ committee members from the remaining $n-1$ students, and there are $\binom{n-1}{k-1}$ ways to do this. In total, we have

$$n\binom{n-1}{k-1} \text{ choices.}$$

Since these two formulas count the same objects, they must be equal.

3. Trinomial Coefficients. Consider integers $i, j, k \geq 0$ such that $i + j + k = n$, and let N be the number of words that can be made with the letters

$$\underbrace{a, a, \ldots, a}_{i \text{ copies}}, \underbrace{b, b, \ldots, b}_{j \text{ copies}}, \underbrace{c, c, \ldots, c}_{k \text{ copies}}.$$

(a) Explain why $n! = N \times i! \times j! \times k!$.
(b) How many words can be made from the letters b, a, n, a, n, a?

(a) Double counting: Suppose that the letters are labeled as

$$a_1, a_2, \ldots, a_i, b_1, b_2, \ldots, b_j, c_1, c_2, \ldots, c_k.$$

Since these n letters are distinct, the number of ways to arrange them is $n!$. On the other hand, suppose that we start with one of the N **unlabeled** arrangements. Then there are $i!$ ways to put labels on the "a"s, $j!$ ways to put labels on the "b"s and $k!$ ways to put labels on the "c"s, for a total of

$$N \times i! \times j! \times k! \text{ arrangements.}$$

(b) If we have $i = 1$ copy of "b," $j = 3$ copies of "n" and $k = 2$ copies of "n," then the number of ways to arrange them is

$$N = \frac{(i+j+k)!}{i! \times j! \times k!} = \frac{6!}{1! \times 2! \times 3!} = 60.$$

Remark: More generally, consider an alphabet $a_1, a_2, \ldots, a_\ell$ of length ℓ. The number of words that can be made containing i_k copies of the letter "a_k" is

$$\frac{(i_1 + i_2 + \cdots + i_\ell)!}{i_1! \times i_2! \times \cdots \times i_\ell!}.$$

4. Falling Factorial. For any number z and for any integer $k \geq 0$ we define the "falling factorial" notation $(z)_k := z(z-1)(z-2)\cdots(z-k+1)$. If $n \geq 0$ is an integer, show that

$$\binom{n}{k} = \frac{(n)_k}{k!}.$$

Pure algebra: If n is a positive whole number then $n!$ exists and we can write

$$\begin{aligned}
\frac{n!}{k! \times (n-k)!} &= \frac{n(n-1)(n-2)\cdots(n-k+1)\cancel{(n-k)(n-k-1)\cdots 3\cdot 2\cdot 1}}{k! \times \cancel{(n-k)(n-k-1)\cdots 3\cdot 2\cdot 1}} \\
&= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \\
&= \frac{(n)_k}{k!}.
\end{aligned}$$

5. Newton's Binomial Theorem. Consider any integer $k \geq 0$. Based on Problem 4, Isaac Newton defined the notation

$$\binom{z}{k} := \frac{(z)_k}{k!}$$

for **any number** z (not just positive whole numbers), and he showed that for any number x with $|x| < 1$ the following infinite series is convergent:

$$(1+x)^z = 1 + \binom{z}{1}x + \binom{z}{2}x^2 + \binom{z}{3}x^3 + \cdots.$$

(a) For any integers $n, k \geq 1$ show that

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}.$$

(b) Use Newton's formula to obtain an infinite series expansion for $(1+x)^{-2}$.

(a) Suppose that n is a positive whole number. Then by definition we have

$$\begin{aligned}
\binom{-n}{k} &= \frac{(-n)_k}{k!} \\
&= \frac{(-n)(-n-1)(-n-2)\cdots(-n-k+1)}{k!} \\
&= \frac{(-1)(n)(-1)(n+1)(-1)(n+2)\cdots(-1)(n+k-1)}{k!} \\
&= \frac{(-1)^k(n+k-1)\cdots(n+2)(n+1)(n)}{k!} \\
&= \frac{(-1)^k(n+k-1)\cdots(n+2)(n+1)(n)\cancel{(n-1)(n-2)\cdots 3\cdot 2\cdot 1}}{k! \times \cancel{(n-1)(n-2)\cdots 3\cdot 2\cdot 1}} \\
&= (-1)^k \cdot \frac{(n+k-1)!}{k!(n-1)!} \\
&= (-1)^k \binom{n+k-1}{k}.
\end{aligned}$$

(b) In the special case $n = 2$ the formula from part (a) gives

$$\binom{-2}{k} = (-1)^k \binom{2+k-1}{k} = (-1)^k \binom{k+1}{k} = (-1)^k (k+1).$$

Then Newton's formula tells us that

$$(1+x)^{-2} = 1 + \binom{-2}{1} x + \binom{-2}{2} x^2 + \binom{-2}{3} x^3 + \cdots$$
$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

Remark: Here's an alternate way to get the same answer. Start with the "geometric series:"

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \cdots .$$

Differentiate both sides by x to get

$$(1-x)^{-2} = 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots$$

Then substitute $x \mapsto -x$ to get

$$(1-(-x))^{-2} = 1 + 2(-x) + 3(-x)^2 + 4(-x)^3 + \cdots$$
$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots .$$