Counting Subsets & Binary Strings

Boolean Algebra is done!
The new topic is the Binomial Theorem.

Last week we discussed how the subsets of

$$U = \{1, 2, 3, \dots, n\}$$

can be encoded as binary strings.
The subset $A \subseteq U$ corresponds to string

$$b_1 b_2 b_3 \dots b_n$$

where the ith "bit" is $b_i = \begin{cases} 1 \text{ if } i \in A \\ 0 \text{ if } i \notin A. \end{cases}$

Example: Here are the subsets of $\{1,2,3\}$ and their corresponding binary strings

$\{1,2,3\}$	111
$\{1,2\}$ $\{1,3\}$ $\{2,3\}$	110 101 011
$\{1\}$ $\{2\}$ $\{3\}$	100 010 001
$\emptyset$	000.

This gives us a convenient way to count subsets.

Theorem: Let U be a set with n elements. Then U has exactly 2^n different subsets.

Proof: This is the same as counting binary strings of length n. There are 2 choices for each bit and there are n independent choices to make.

Thus the total number of choices is

$$\underset{1st}{\underline{2}} \times \underset{2nd}{\underline{2}} \times \underset{3rd}{\underline{2}} \times \ldots \times \underset{nth}{\underline{2}} = 2^n$$

///

This week we are interested in a more refined problem.

<u>Problem</u>: Let U be a set with n elements and consider an integer k such that $0 \leq k \leq n$. Then <u>how many</u> subsets with k elements does U have?

Equivalently, how many of the 2^n binary strings of length n contain exactly k 1's (and hence $n-k$ 0's)?

<u>Example</u>

111 ——— 1

110 101 011 ——— 3

100 010 001 ——— 3

000 ——— 1.

We are interested in this equation:

$$2^3 = 1 + 3 + 3 + 1 .$$

$$2^n = \text{what ?}$$

To solve this we will need a "preliminary fact" (which we call a "lemma").

Q: Given n different symbols, in how many ways can I write them in a line?

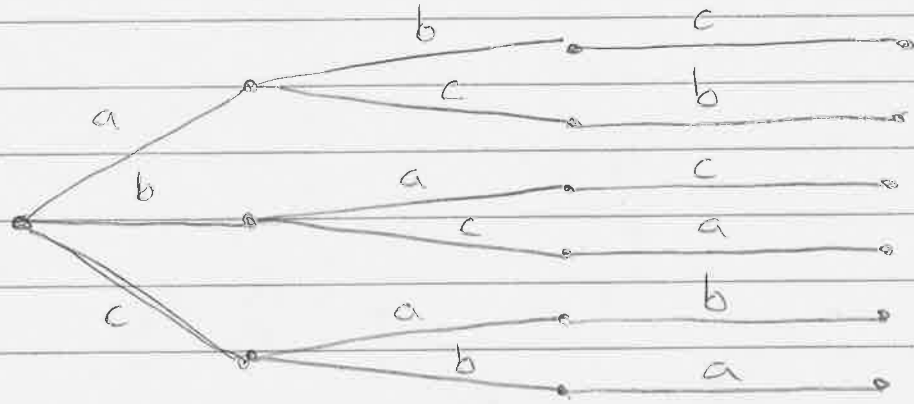
Example: Using symbols a, b, c the possibilities are:

abc, acb, bac, bca, cab, cba.

We call these the permutations of the symbols. There are 6 of them.

Q: How many permutation are there of n different symbols?

First let's note that $6 = 3 \times 2 \times 1$. We can arrange the permutations of a, b, c in a tree like this:



So we really want to count the branches of this tree. The total # of branches is

$$\underset{1st}{\underline{3}} \times \underset{2nd}{\underline{2}} \times \underset{3rd}{\underline{1}} = 6$$

In general, given a positive integer n we define the notation

$$n! := n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$$

We call this "n factorial".

Lemma: The number of permutations of n different symbols is $n!$

Proof: There are n ways to choose the first/leftmost symbol. Then there are $n-1$ remaining choices for the 2nd symbol. Continuing in this way, the total number of choices is

$$\underset{1st}{\underline{n}} \times \underset{2nd}{\underline{n-1}} \times \underset{3rd}{\underline{n-2}} \times \cdots \times \underset{(n-1)^{th}}{\underline{2}} \times \underset{n^{th}}{\underline{1}} = n!$$

//

These numbers grow fast!

n	1	2	3	4	5	6	7	$\cdots$
$n!$	1	2	6	24	120	720	5040	$\cdots$

James Stirling (1692 – 1770) gave a charming and surprising formula for their rate of growth. He proved

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

"Stirling's Approximation".

That is surprising right?

However, the problem of permutations is not so easy when some of the symbols are the same.

Example: How many permutations of

a, a, b, b ?

aabb, abab, abba, baab, baba, bbaa

Answer: It's not $4! = 24$. It's just 6.

Example: How many permutations of

a, a, a, b, b, b, b ?

Now it's too many to do by hand. We need to think about it systematically somehow...

We need a trick.

Here's the trick: Let's temporarily label the symbols.

$$a_1, a_2, a_3, b_1, b_2, b_3, b_4.$$

Now we know that there are

$$7! = 5040$$

permutations. But this number is too big because many of these correspond to the same <u>unlabeled</u> permutations.

For example, the <u>labeled</u> permutations

$$a_3 b_4 a_1 b_1 b_3 a_2 b_2 \ \& \ a_1 b_3 a_3 b_1 b_2 a_2 b_4$$

both correspond to the <u>unlabeled</u> permutation

$$ababbab.$$

We need to find out exactly how often this happens.

For example, how many times does ababbab show up in our count of 5040?

Actually this not too hard. There are $3! = 6$ ways to label the a's and $4! = 24$ ways to label the b's.

ababbab → $a_1 b_1 a_2 b_2 b_3 a_3 b_4$

ababbab → $a_1 b_1 a_3 b_2 b_3 a_2 b_4$

etc.

$$3! \times 4! = 6 \times 24$$
$$= 144.$$

In fact, every unlabeled permutation will get counted 144 times because because there is always 144 ways to label it.

Now we can solve the problem.

Let N = The number of unlabeled permutations of a, a, a, b, b, b, b. Since each of these got counted 144 times, we conclude that

$$7! = N \times 3! \times 4!$$
$$5040 = 144 \cdot N$$

$$\Longrightarrow N = \frac{5040}{144} = 35 .$$

(I guess we could have counted those by hand, but it would have taken a while, and we probably would have made mistakes.)

Now we will use the same trick to solve the general problem.

Theorem: Let U be a set with n elements and let k be an integer such that $0 \leq k \leq n$.

$\int$

Then the number of subsets of $\mathcal{U}$ with k elements is given by

$$\frac{n!}{k!\,(n-k)!}$$

Proof: This is the same as counting permutations of the symbols

$$\underbrace{1, 1, 1, \ldots, 1}_{k \text{ of these}}, \underbrace{0, 0, 0, \ldots, 0}_{n-k \text{ of these}}$$

i.e., binary strings of length n containing k 1's and $n-k$ 0's.
Let N be the number of such strings.
We want to find an equation for N.

To do this we will consider an auxiliary problem, to count the permutations of the labeled symbols

$$1_1, 1_2, 1_3, \ldots, 1_k, 0_1, 0_2, \ldots, 0_{n-k}$$

On one hand there are $n!$ such permutations because these n symbols are all different.

On the other hand, these labeled permutations break up into groups corresponding to the different unlabeled permutations. Each of these groups has the same size $k!(n-k)!$, because given any unlabeled permutation there are $k!(n-k)!$ ways to label it. [$k!$ ways to label the 1's and $(n-k)!$ ways to label the 0's.]

We conclude that

$$\underset{\substack{\text{order the}\\ \text{labeled symbols}}}{n!} = \underset{\substack{\text{order the}\\ \text{unlabeled symbols}}}{N} \times \underbrace{k! \times (n-k)!}_{\text{label them}}$$

Hence

$$N = \frac{n!}{k!(n-k)!}$$

Some Remarks:

- It is not obvious that $n!/(k!(n-k)!)$ is even an <u>integer</u>, but we just proved that it is, because it <u>counts</u> something.

- The method we used is called "double counting": Count a certain set in two different ways to get an equation. It is very useful.

- Wait! Is our formula true when $k=0$ or $k=n$?

Hmm...

It depends what you mean by $0!$

Binomial Theorem

Current Topic : Binomial Theorem.

Last time you proved the following.

☆ Theorem: Let U be a set with n elements and let k be an integer such that $0 \le k \le n$. The number of subsets of U of size k is equal to

$$\frac{n!}{k!(n-k)!}$$

We will use a special notation for these numbers

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

"n choose k".

Example: I have 10 (different) books on my shelf and I want to give you 4 of them. In how many ways can I do this?

$$\binom{10}{4} = \frac{10!}{4!\,6!}$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot \cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}$$

$$= \frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8} \cdot 7}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 10 \cdot 3 \cdot 7 = 210.$$

///

Discussion:

- We observe that $\binom{n}{k} = \binom{n}{n-k}$. There are two ways to see this.

i) Directly from the formula:

$$\binom{n}{n-k} = \frac{n!}{(n-k)!\,(n-(n-k))!} = \frac{n!}{(n-k)!\,k!}$$

$$= \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

///

ii) Using a counting argument. Let $\# U = n$. Let $X \subseteq \wp(U)$ be the set of subsets containing k elements and let $Y \subseteq \wp(U)$ be the set of subsets containing $n-k$ elements. Then we have a bijection

$$\begin{array}{ccc} X & \longleftrightarrow & Y \\ A & \longmapsto & A^c \\ B^c & \longleftarrow\!\shortmid & B \end{array}$$

By HW2.1c we conclude that

$$\binom{n}{k} = \# X = \# Y = \binom{n}{n-k}.$$

Example: $U = \{1, 2, 3\}$, $k = 1$.

$$\begin{array}{ccc} \{1\} & \longleftrightarrow & \{2,3\} \\ \{2\} & \longleftrightarrow & \{1,3\} \\ \{3\} & \longleftrightarrow & \{1,2\} \end{array}$$

$\binom{3}{1} = 3$ $\qquad$ $\binom{3}{2} = 3$.

///

- We know that $\binom{n}{0} = \binom{n}{n} = 1$, but does this agree with the formula?

$$\binom{n}{0} = \frac{n!}{0!\,n!} = \frac{1}{0!} = ?$$

Wait! What is $0!$?

$$0! = 0 \cdot (-1) \cdot (-2) \cdot (-3) \cdot (-4) \cdot \cdots$$

That makes no sense. So we will just define it to be

$$0! := 1.$$

Ok? Then

$$\binom{n}{0} = \binom{n}{n} = \frac{1}{0!} = 1$$

as desired. This is mostly a notational convenience, but there are deeper reasons (involving the "gamma function"

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \quad).$$

- Since the total number of subsets of U is 2^n we have a nice equation

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

Examples:

$$1 = 1 \qquad \left(2^0 = \binom{0}{0}\right)$$
$$2 = 1 + 1$$
$$4 = 1 + 2 + 1$$
$$8 = 1 + 3 + 3 + 1$$
$$16 = 1 + 4 + 6 + 4 + 1 .$$

Actually, this is just the shadow of a more interesting equation. Let a, b be <u>any</u> numbers and consider the number

$$(a+b)^n .$$

$$(a+b)^0 = 1$$
$$(a+b)^1 = a + b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Do you recognize this ?

The official statement is called

☆ The Binomial Theorem: For all numbers a & b and for all integers $n \geq 0$ we have

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

How can we prove this?

Proof: Let's temporarily pretend that $ab \neq ba$.

Then we have, for example,

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b) \\ &= a(a+b) + b(a+b) \\ &= aa + ab + ba + bb \\ &= aa + \begin{pmatrix} ab \\ + \\ ba \end{pmatrix} + bb\end{aligned}$$

$$(1 \quad 2 \quad 1)$$

and $(a+b)^3 = (a+b)(a+b)^2$
$= (a+b)(aa + ab + ba + bb)$

$$= aaa + aab + aba + abb$$
$$+ baa + bab + bba + bbb$$

$$= aaa + \begin{pmatrix} aab \\ + \\ aba \\ + \\ baa \end{pmatrix} + \begin{pmatrix} abb \\ + \\ bab \\ + \\ bba \end{pmatrix} + bbb.$$

$$(\underline{1} \qquad 3 \qquad 3 \qquad 1).$$

In general, we see that $(a+b)^n$ is the sum of all words of length n using the letters a & b. We know that the number of such words containing k a's and $n-k$ b's equals $\binom{n}{k}$. [How do we know this?]

Thus if we allow $ab = ba$ then the term $a^k b^{n-k}$ will occur $\binom{n}{k}$ times in the exponsion of $(a+b)^n$. In other words,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

[Recall how we counted the words with k a's and $n-k$ b's. Let N be the number of such words. We will count permutations of the symbols

$$a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_{n-k}$$

in two different ways:

$$\underset{\substack{\uparrow \\ \#\text{ labeled} \\ \text{words}}}{n!} = \underset{\substack{\uparrow \\ \#\text{ unlabeled} \\ \text{words}}}{N} \cdot \underbrace{k! \cdot (n-k)!}_{\substack{\uparrow \\ \#\text{ ways to} \\ \text{Label them.}}}$$

]

Q: What good is the Binomial Theorem?

A: Well, it remains true for any values of a & b that we substitute.

For example, if we put $a = 1$ and $b = -1$ then we get

$$(1-1)^n = \sum_{k=0}^{n} \binom{n}{k} (1)^k (-1)^{n-k}$$

$$= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}.$$

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n}$$

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

even subsets = # odd subsets

[Wait! What about when $n = 0$?]

We can even treat $(a+b)^n$ as a function of a & b and do things like differentiate it.

Example: Let $a = x$ and $b = 1$, so

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$(1+x)^n = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n.$$

Now differentiate both sides by x:

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \cdots + n\binom{n}{n}x^{n-1}$$

Now substitute $x=1$ to get

$$n \cdot 2^{n-1} = 1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}$$

That was pure algebra. On the other hand this equation has on interpretation in terms of counting subsets.

Problem: Let U be a set of n people. Find the number of ways of choosing a committee with a president.

One one hand we con choose the president first in n ways and then we can choose the other committee members in 2^{n-1} ways, for a total of

$$n \cdot 2^{n-1} \quad \text{choices.}$$

choose president

choose the other $n-1$ committee members.

On the other hand, we could choose the committee first and then the president. If the committe has k members then the number of choices is

$$\underbrace{\binom{n}{k}}_{\text{choose the committee}} \cdot \underbrace{k}_{\text{then choose the president from the committee.}}$$

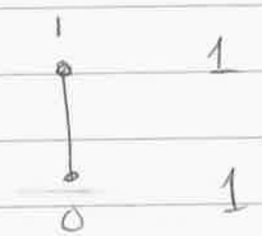
To get the total number of choices we sum over all possible sizes of committee:

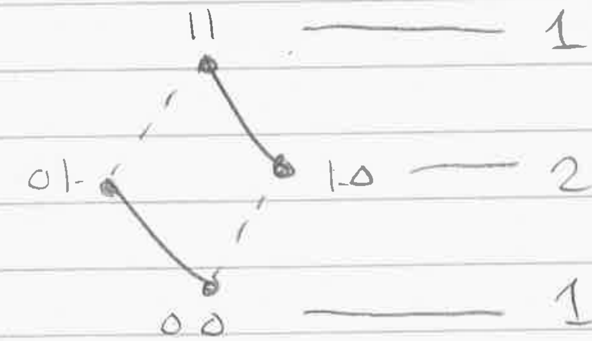
$$1 \cdot \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}.$$

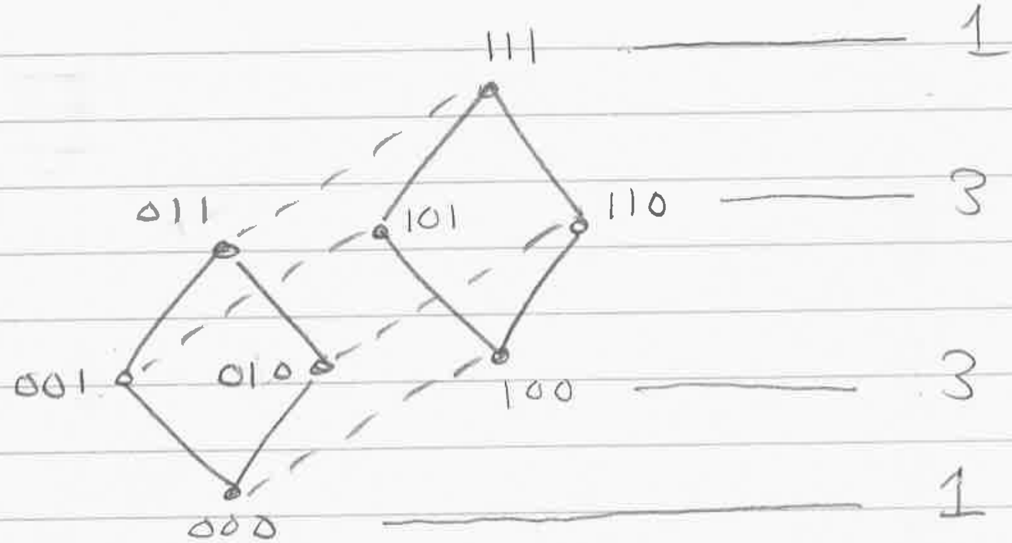
///

Which proof do you like better: the counting proof or the algebra?

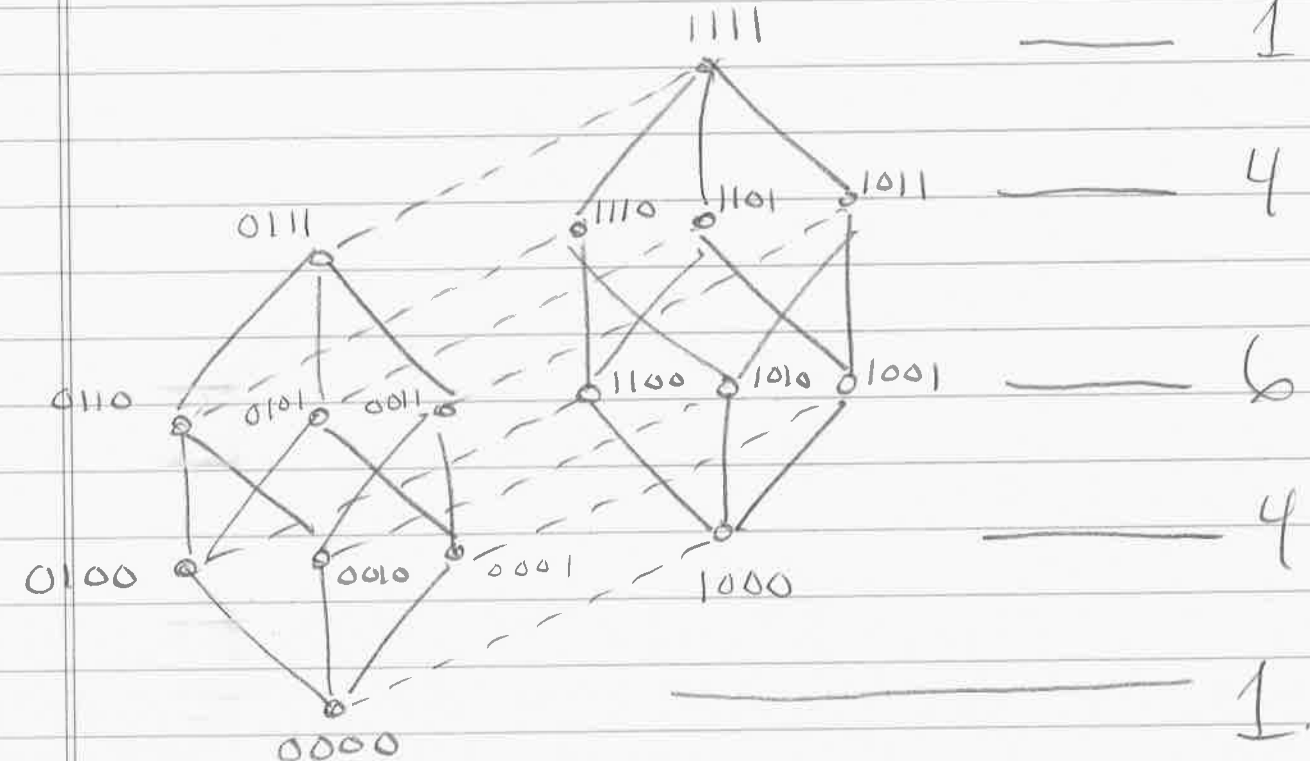
Next Wednesday we will discuss the recursive structure of the Binomial Theorem. It has to do with the recursive structure of cubes.







4 dimensional cube:



and so on.

Pascal's Triangle

Today : Pascal's Triangle.

We have seen that the numbers $\binom{n}{k}$ (read "n choose k") have several interpretations.

- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

- $\binom{n}{k}$ = The number of ways to choose k unordered things from a set of n unordered things.

- $\binom{n}{k}$ = The number of binary strings with k 1's and $n-k$ 0's.

- The Binomial Theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

The fact that these four interpretations are equivalent needs to be proved, which we proved in the previous two classes. Once we know this fact we can apply it.

<u>Example</u>: How many "words" can you make using all of the letters

$$a, a, a, b, b, b, b \quad ?$$

Answer: $\binom{7}{3} = \binom{7}{4} = \dfrac{7!}{3!\,4!}$

$$= \frac{7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}{\cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 35$$

[Do you remember how to prove this? Count the words you can make from

$$a_1, a_2, a_3, b_1, b_2, b_3, b_4$$

in two different ways:

$$7! = X \cdot 3! \cdot 4!$$
]

Example: How many subsets of size 3 does the set $\{1,2,3,4,5,6,7\}$ have?

Answer: 35 again, because its the same problem. There is a bijection between the words and the subsets.

$$\{2,4,5\} \subseteq \{1,2,\dots,7\} \Longleftrightarrow \underset{(1\,2\,3\,4\,5\,6\,7)}{babaabb}.$$

Example: Expand $(a+b)^7$ by hand.

Answer:

$$(a+b)^7 = \binom{7}{0}b^7 + \binom{7}{1}ab^6 + \binom{7}{2}a^2b^5 + \binom{7}{3}a^3b$$

$$+ \binom{7}{4}a^4b^3 + \binom{7}{5}a^5b^2 + \binom{7}{6}a^6b + \binom{7}{7}a^7.$$

But it could take a few minutes to compute these coefficients. I'll show you a helpful shortcut.

They are just the entries in the 7^{th} row of "Pascal's Triangle"

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

So the answer is

$$(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3$$
$$+ 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7.$$

Why does that work? Let's figure out first exactly what it is that is working. We claim that the k^{th} entry of the n^{th} row of Pascal's Triangle equals

$$\binom{n}{k}.$$

That is,

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

etc.

Pascal's Triangle is <u>defined</u> by the fact that each entry is the sum of the two above.

$$\begin{array}{ccc} \binom{n-1}{k-1} & & \binom{n-1}{k} \\ & \searrow \quad \swarrow & \\ & \binom{n}{k} & \end{array}$$

So we need to prove that for all relevant values of n and k we have

$$\boxed{\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.}$$

We can use <u>any</u> of the different interpretations to prove this. You'll give two different proofs on HW 4. Here is a third proof, using the Binomial Theorem.

☆ Theorem : $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

<u>Proof</u> : We will take for granted the fact that for all numbers a, b and for all integers $n \geq 0$ we have

$$(a+b)^n = \binom{n}{0} b^n + \binom{n}{1} a b^{n-1} + \cdots + \binom{n}{n-1} a^{n-1} b + \binom{n}{n} a^n.$$

Then we will use a <u>very</u> <u>small</u> <u>trick</u> :

$$\begin{aligned}(a+b)^n &= (a+b)(a+b)^{n-1} \\ &= a(a+b)^{n-1} + b(a+b)^{n-1}.\end{aligned}$$

Now we just put everything together :

(It won't fit here. Turn the page.)

$$\binom{n}{0} b^n + \binom{n}{1} a b^{n-1} + \cdots + \binom{n}{k} a^k b^{n-k} + \cdots + \binom{n}{n} b^n.$$

$$= a\left[\binom{n-1}{0} b^{n-1} + \cdots + \binom{n-1}{n-1} a^{n-1}\right]$$

$$+ b\left[\binom{n-1}{0} b^{n-1} + \cdots + \binom{n-1}{n-1} a^{n-1}\right]$$

$$= \left[0 + \binom{n-1}{0} a b^{n-1} + \binom{n-1}{1} a^2 b^{n-1} + \cdots + \binom{n-1}{n-2} a^{n-1} b + \binom{n-1}{n-1} a^n\right]$$

$$+ \left[\binom{n-1}{0} b^n + \binom{n-1}{1} a b^{n-1} + \binom{n-1}{2} a^2 b^{n-1} + \cdots + \binom{n-1}{n-1} a^{n-1} b + 0\right]$$

$$= \left(0 + \binom{n-1}{0}\right) b^n + \left(\binom{n-1}{0} + \binom{n-1}{1}\right) a b^{n-1} + \cdots$$

$$\cdots + \left(\binom{n-1}{k-1} + \binom{n-1}{k}\right) a^k b^{n-k} + \cdots$$

$$\cdots + \left(\binom{n-1}{n-2} + \binom{n-1}{n-1}\right) a^{n-1} b + \left(\binom{n-1}{n-1} + 0\right) a^n.$$

Comparing the coefficient of $a^k b^{n-k}$ on both sides gives

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Remark: For the equation to work at the ends we need to say that

$$\binom{n}{-1} = \binom{n}{n+1} = 0.$$

We will say this. In fact, for all $n \geq 0$ we will say that

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{if } 0 \leq k \leq n. \\ 0 & \text{otherwise} \end{cases}$$

This shows that the binomial coefficients $\binom{n}{k}$ are the same as the entries of P.T.

$$\begin{array}{c}
\cdots\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ \cdots \\
\cdots\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ \cdots \\
\cdots\ 0\ 0\ 0\ 0\ 1\ 2\ 1\ 0\ 0\ 0\ 0\ \cdots \\
\cdots\ 0\ 0\ 0\ 1\ 3\ 3\ 1\ 0\ 0\ 0\ \cdots \\
\cdots\ 0\ 0\ 1\ 4\ 6\ 4\ 1\ 0\ 0\ \cdots \\
\cdots\ 0\ 1\ 5\ 10\ 10\ 5\ 1\ 0\ \cdots \\
\cdots\ 1\ 6\ 15\ 20\ 15\ 6\ 1\ \cdots
\end{array}$$

I could have phrased this in a different way. I could have asked you to <u>solve</u> the following recurrence and initial conditions:

$$\boxed{\begin{array}{l} \bullet \ f(0,k) = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases} \\ \\ \bullet \ f(n,k) = f(n-1,k) + f(n-1,k-1) \quad \forall n,k \in \mathbb{Z}, n \geq 1. \end{array}}$$

Wow, this is more complicated than previous recusion problems, but we already know the answer.

<u>Theorem</u>: The solution is $f(n,k) = \binom{n}{k}$.

Proof by induction on n:

(1) First we verify the base case. Indeed, we know that

$$\binom{0}{k} = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases}$$

hence $f(0,k) = \binom{0}{k}$ for all $k \in \mathbb{Z}$.

(2) Now fix some $n \geq 1$ and assume for induction that we have

$$f(n,k) = \binom{n}{k} \quad \forall\, k \in \mathbb{Z}.$$

In this hypothetical case we want to prove that

$$f(n+1,k) = \binom{n+1}{k} \quad \forall\, k \in \mathbb{Z}.$$

Indeed, for all $k \in \mathbb{Z}$ we have

$$\begin{aligned} f(n+1,k) &= f(n,k) + f(n,k-1) && \text{by definition} \\ &= \binom{n}{k} + \binom{n}{k-1} && \text{by induction hypothesis} \\ &= \binom{n+1}{k} && \text{by the Theorem proved earlier in today's class.} \end{aligned}$$

By induction we conclude that we have

$$f(n,k) = \binom{n}{k} \quad \forall\, k \in \mathbb{Z}.$$

for all $n \geq 0$.

Remark: I don't expect you to solve general recurrences that complicated. This is a very special case.

In class we used the formula n! / k!(n-k)! to give a different proof of the recurrence

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

And here's an idea for yet another proof:

Consider the set of binary strings with k 1's and $n-k$ 0's. Divide them into two sets based on their leftmost bit.

Example: $n=5$, $k=3$.

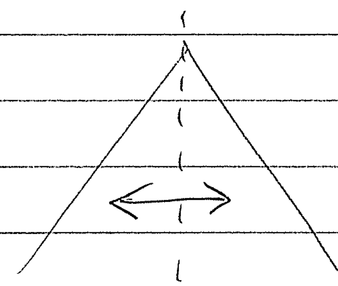
11100	01110
11010	01101
11001	01011
10110	00111
10101	
10011	

$$6 \quad + \quad 4 \quad = \quad 10$$

What do you see?

Remarks:

1. The symmetry $\binom{n}{k} = \binom{n}{n-k}$ is evident in the picture.



left-right symmetry

2. We conventionally define

$$\binom{n}{k} = 0 \quad \text{if} \quad k < 0 \quad \text{or} \quad n < k .$$

$$\begin{array}{ccccccccccccccccccc}
\cdots & 0 & & 0 & & 0 & & 1 & & 0 & & 0 & & 0 & \cdots \\
\cdots & & 0 & & 0 & & 1 & & 1 & & 0 & & 0 & & \cdots \\
\cdots & 0 & & 0 & & 1 & & 2 & & 1 & & 0 & & 0 & \cdots \\
\cdots & & 0 & & 1 & & 3 & & 3 & & 1 & & 0 & & \cdots
\end{array}$$

3. But what if <u>n</u> is negative?

Does Pascal's Triangle go <u>up</u>?

Let us adopt the boundary conditions

$$\binom{-n}{0} = 1 \quad \text{for all integers } n.$$

Then we can use Pascal's recurrence to extend the triangle <u>up</u>:

$$\begin{array}{cccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 1 & & -4 & & +10 \\
0 & & 0 & & 0 & & 0 & & 1 & & -3 & & +6 & \\
 & 0 & & 0 & & 0 & & 1 & & -2 & & +3 & & -4 \\
0 & & 0 & & 0 & & 1 & & -1 & & +1 & & -1 & \\
\hline
 & 0 & & 0 & & 1 & & 0 & & 0 & & 0 & & 0 \\
0 & & 0 & & 1 & & 1 & & 0 & & 0 & & 0 & \\
 & 0 & & 1 & & 2 & & 1 & & 0 & & 0 & & 0 \\
0 & & 1 & & 3 & & 3 & & 1 & & 0 & & 0 & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & & 0 & & 0
\end{array}$$

Check: $\begin{array}{ccc} -3 & & +6 \\ & +3 & \end{array}$ $\qquad -3+6=+3$ ✓

On HW3 you will prove the formula

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$$

and explore its meaning.

Methods of Counting

Four Ways to Count:

Let S be a set with n elements. By definition of "set" these elements are

- unordered (there is no "1st" element...)
- distinct (no repeated elements).

Now we want to "choose" k elements from S. There are at least 4 ways to interpret the word "choose".

We might grab all k elements at once or we might pick them one at a time. If we pick them one at a time then we might/might not

- replace each element after we pick it, so we might get it multiple times.
- record the order in which we picked the elements.

We can arrange this information in a table. If we "choose" k things from a set of n, the number of possibilities is given by

	order matters	order irrelevant
with replacement	n^k	?
without replacement	$\binom{n}{k} \cdot k!$	$\binom{n}{k}$

These three entries are pretty straightforward once we know about binomial coefficients.

Bottom-Right: We've been talking about this all week.

Top-Left: We can think of S as a set of symbols (an "alphabet") Then we want to count the number of words of length k that we can make from this alphabet. (Order of symbols matters and symbols can be repeated.) The answer is

$$\underset{\text{1st letter}}{\underline{\overset{n}{\text{choices}}}} \times \underset{\text{2nd letter}}{\underline{\overset{n}{\text{choices}}}} \times \cdots \times \underset{k^{th}\text{ letter}}{\underline{\overset{n}{\text{choices}}}} = n^k$$

Bottom-Left: To choose an ordered set of k elements without repetition, first choose an unordered set of k elements in $\binom{n}{k}$ ways, then order this set in $k!$ ways.

The total number of possibilities is

$$\underbrace{\binom{n}{k}}_{\text{choose an unordered set}} \cdot \underbrace{k!}_{\text{order it.}}$$

Top-Right : ?

This is the hardest one. It requires a clever trick.

First let's do an example. Choose 2 elements from $\{a, b, c\}$ with repetition, but don't record the order. The choices are

$$\begin{array}{lll} a, a & b, b & c, c. \\ a, b & b, c & \\ a, c & & \end{array}$$

Why are there 6 ?

I'll show you a bijection to certain kinds of binary strings:

$$\begin{array}{lcl} a, a & \longrightarrow & 0011 \\ a, b & \longrightarrow & 0101 \\ a, c & \longrightarrow & 0110 \\ b, b & \longrightarrow & 1001 \\ b, c & \longrightarrow & 1010 \\ c, c & \longrightarrow & 1100 \end{array}$$

How does that work?

Idea: The 0's are "things" and the 1's are "dividers". That is

some # 0's representing a's	1	some # 0's representing b's	1	some # 0's representing c's

$$\begin{array}{ccccc} 00 & | & 000 & | & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \text{two a's} & & \text{3 b's} & & \text{1 c} \end{array}$$

Clever right? (I didn't invent it.)

To encode a choice of k things from n we will use

k 0's (things)
$n-1$ 1's (dividing lines)

So the total # choices is the number of binary strings with k 0's and $n-1$ 1's, which we know to be

$$\binom{k+(n-1)}{k}.$$

The completed table of ways to count is

	ordered	NOT ordered	
	n^k	$\binom{n+k-1}{k}$	with replacement
	$\binom{n}{k} \cdot k!$	$\binom{n}{k}$	without replacement.

Thinking Problem:

Is there some mystical significance to the equation

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k} \quad ?$$

"Choosing negative things without replacement is the same as choosing positive things with replacement."

What ?!

Application: Coin Flipping

Recall the

☆ Binomial Theorem: For all real or complex numbers x, y and for all positive integers n we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$= x^n + n x^{n-1} y + \frac{n(n-1)}{2} x^{n-2} y^2 + \cdots + \frac{n(n-1)}{2} x^2 y^{n-2} + n x y^{n-1} + y^n$$

Today we will apply this. The most important applications are in the theory of <u>probability</u>.

Let p and q be positive real numbers such that

$$p + q = 1$$

Then the Binomial Theorem says:

$$1 = 1^n = (p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$$

This has the following interpretation. Suppose you have a biased coin such that

$$\text{Prob}(\text{"heads"}) = p$$
$$\text{Prob}(\text{"tails"}) = q$$

If you flip the coin n times, what is the probability that you get "heads" exactly k times?

Example: If you flip the coin 3 times the probability of getting the sequence HTH is

$$\begin{aligned}\text{Prob}(HTH) &= \text{Prob}(H)\,\text{Prob}(T)\,\text{Prob}(H) \\ &= pqp \\ &= p^2 q.\end{aligned}$$

To compute the probability of getting exactly 2 heads, we sum over the ways it can happen.

Prob (getting heads twice)

$= \text{Prob}(\{HHT, HTH, THH\})$

$= \text{Prob}(HHT) + \text{Prob}(HTH) + \text{Prob}(THH)$

$= p^2 q + p^2 q + p^2 q$

$= 3p^2 q.$

In general, the probability of getting exactly k heads in n tosses is

$$\binom{n}{k} p^k q^{n-k}$$

Example: Suppose in a certain population each birth has

$$\text{Prob}(\text{boy}) = 1/3 \qquad \left(\frac{1}{3} + \frac{2}{3} = 1\right)$$
$$\text{Prob}(\text{girl}) = 2/3$$

If a certain family has 4 children, how many boys are they likely to have?

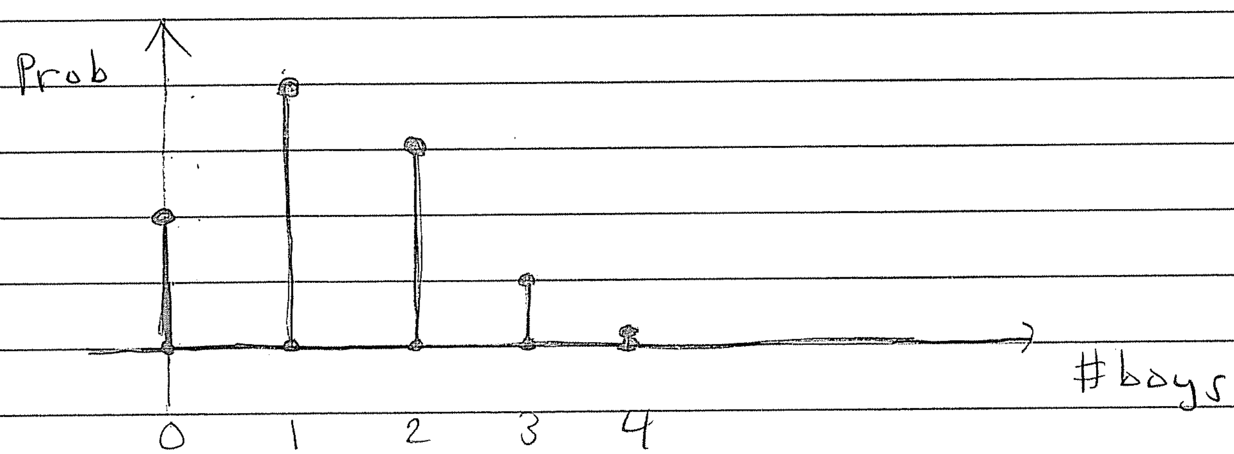
We'll compute the full distribution

# boys	0	1	2	3	4
Prob	$\binom{4}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^4$	$\binom{4}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^3$	$\binom{4}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2$	$\binom{4}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^1$	$\binom{4}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^0$
	$1 \cdot \frac{1^0 \cdot 2^4}{3^4}$	$4 \cdot \frac{1^1 \cdot 2^3}{3^4}$	$6 \cdot \frac{1^2 \cdot 2^2}{3^4}$	$4 \cdot \frac{1^3 \cdot 2^1}{3^4}$	$1 \cdot \frac{1^4 \cdot 2^0}{3^4}$
	$\frac{16}{81}$	$\frac{32}{81}$	$\frac{24}{81}$	$\frac{8}{81}$	$\frac{1}{81}$

Notice that

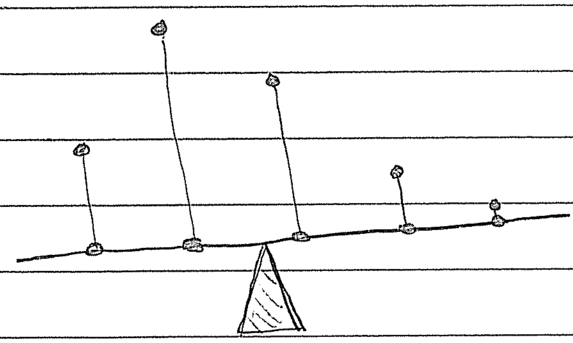
$$\frac{16}{81} + \frac{32}{81} + \frac{24}{81} + \frac{8}{81} + \frac{1}{81} = \frac{81}{81} = 1$$

as it should be. We can think of probability as a distribution of mass



Then the "average" or "expected outcome" is the same as the "center of mass".

Where is it? Where does it balance?



Archimedes tells us the answer. His "law of the lever" says that

mass $\times$ (distance from center)

should balance. In our case, the "center of mass" is

$$0 \cdot \frac{16}{81} + 1 \cdot \frac{32}{81} + 2 \cdot \frac{24}{81} + 3 \cdot \frac{8}{81} + 4 \cdot \frac{1}{81}$$

$$= \frac{0 + 32 + 48 + 24 + 4}{81} = \frac{108}{81} = \frac{4}{3}$$

$$= 1.33333\ldots$$

In a family with 4 children we expect

1.333... boys.

Is that surprising? NO. It's just

$$\underset{\text{Prob(boy)}}{p} \cdot \underset{\text{\# of children}}{n}$$

Theorem: Flip a biased coin ($\text{Prob}(H) = p$, $\text{Prob}(T) = q$, $p + q = 1$) n times. The expected number of heads is

$$pn$$

Proof: By Archimedes' principle, the expected number of heads is

$$E(\#\text{ heads}) = \sum_{k=0}^{n} \underbrace{k}_{\text{"distance"}} \cdot \underbrace{\text{Prob}(k \text{ heads})}_{\text{"mass"}}$$

Since Prob(k heads) $= \binom{n}{k} p^k q^{n-k}$ we have

$$E(\#\text{ heads}) = \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k}$$

What tricks do we have for evaluating this sum? Here's the key trick:

$$k \binom{n}{k} = \frac{k \, n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$$

$$= n \cdot \frac{(n-1)!}{(k-1)!(n-k)!} = n \cdot \frac{(n-1)!}{(k-1)!\,[(n-1)-(k-1)]!}$$

$$= n \binom{n-1}{k-1}$$

In summary, we have

$$k \binom{n}{k} = n \binom{n-1}{k-1}.$$

Plugging this in to our sum gives

$$E(\#\text{ heads}) = \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=1}^{n} n \binom{n-1}{k-1} p^{k} q^{n-k}$$

$$= pn \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} q^{n-k}$$

$$= pn \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} q^{(n-1)-(k-1)}$$

$$= pn \sum_{j=0}^{n-1} \binom{n-1}{j} p^{j} q^{(n-1)-j}$$

$$= pn \left(p + q \right)^{n-1}$$

$$= pn \left(1 \right)^{n-1}$$

$$= pn.$$

//

There may be trickier tricks, but that one let us practice our skills.

Thinking Problem: What is the expected value of

$$(\# \text{heads} - pn)^2 \quad ?$$