Math 309 — **Summer B 2018**
Homework 2 — **Drew Armstrong**

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1. De Morgan's Laws.

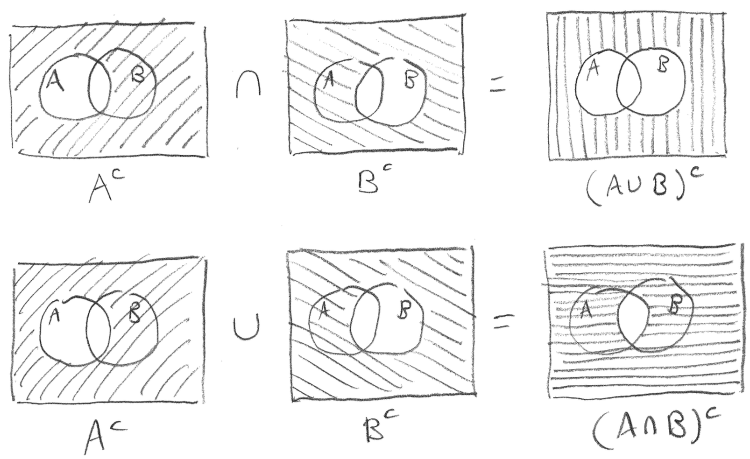
(a) Let $A, B \subseteq U$ be any subsets of the universal set. Use Venn diagrams to show that

$$(A \cup B)^c = A^c \cap B^c \qquad \text{and} \qquad (A \cap B)^c = A^c \cup B^c.$$

(b) Let P, Q be any logical statements. Use truth tables to show that

$$\neg(P \vee Q) = \neg P \wedge \neg Q \qquad \text{and} \qquad \neg(P \wedge Q) = \neg P \vee \neg Q.$$

(a) Here are the Venn diagrams:



(b) Observe that the 4th and 7th columns in each truth table are are equal:

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

2. The Contrapositive. Let P and Q be logical statements. We define the statement $P \Rightarrow Q$ (read as "P implies Q") by the formula

$$P \Rightarrow Q := (\neg P) \vee Q = (\text{NOT } P)\, \text{OR}\, Q.$$

(a) Draw the truth table for this function.
(b) Use a truth table or another method to show that "$P \Rightarrow Q$" is the same as "$\neg Q \Rightarrow \neg P$."
(c) If R is another logical statment, use Problem 1 and part (b) to show that

$$\text{“}P \Rightarrow (Q \vee R)\text{”} = \text{“}(\neg Q \wedge \neg R) \Rightarrow \neg P.\text{”}$$

(a)

P	Q	$\neg P$	$(\neg P)\vee Q$	$P\Rightarrow Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

(b) Since $\vee$ is commutative we have

$$\text{“}\neg Q\Rightarrow\neg P\text{”} = \text{“}(\neg\neg Q)\vee(\neg P)\text{”} = \text{“}Q\vee(\neg P)\text{”} = \text{“}(\neg P)\vee Q\text{”} = \text{“}P\Rightarrow Q\text{”}$$

(c) By applying Problem 2(b) and Problem 1(b) we obtain

$$\begin{aligned} \text{“}P\Rightarrow(Q\vee R)\text{”} &= \text{“}\neg(Q\vee R)\Rightarrow\neg P\text{”} && 2(b) \\ &= \text{“}(\neg Q\wedge\neg R)\Rightarrow\neg P\text{”} && 1(b) \end{aligned}$$

3. Application. We way that an integer $n\in\mathbb{Z}$ is **odd** when $n=2k+1$ for some $k\in\mathbb{Z}$. We that n is **even** when it is not odd. Consider any two integers $m,n\in\mathbb{Z}$ and use Problem 2(c) to prove the following statement:

“If mn is even then m is even or n is even.”

[Hint: Let $P=$ “mn is even,” $Q=$ “m is even,” and $R=$ “n is even.”]

We are asked to prove that $P\Rightarrow(Q\vee R)$ is a true statement. By Problem 2(c), this statement is logically equivalent to $(\neg Q\wedge\neg R)\Rightarrow\neg P$, which in English says:

“If m and n are both odd then mn is odd.”

In order to prove this equivalent statement, let us suppose that m and n are both odd. This means that $m=2k+1$ and $n=2\ell+1$ for some integers $k,\ell\in\mathbb{Z}$. But then we have

$$\begin{aligned} mn &= (2k+1)(2\ell+1) \\ &= 4k\ell+2k+2\ell+1 \\ &= 2(2k\ell+k+\ell)+1 \\ &= 2(\text{something})+1, \end{aligned}$$

and it follows that mn is odd. This completes the proof. □

4. Counting Functions. Let S be a finite set with $\#S$ elements and let T be a finite set with $\#T$ elements.

(a) Write a formula for the number of elements of $S\times T$, i.e., the number of ordered pairs (s,t) with $s\in S$ and $t\in T$.
(b) Write a formula for the number of functions from S to T.
(c) Use your answers from parts (a) and (b) to compute the number of Boolean functions with 2 inputs and 1 output. [Hint: By definition these are the functions from $\{T,F\}\times\{T,F\}$ to $\{T,F\}$.]

(a) Suppose that $\#S=m$ and $\#T$, and let us denote the elements as

$$S=\{s_1,s_2,\ldots,s_n\} \qquad \text{and} \qquad T=\{t_1,t_2,\ldots,t_n\}.$$

Then the elements of the set $S \times T$ can be arranged in a rectangle as follows:

	t_1	t_2	$\cdots$	t_n
s_1	(s_1, t_1)	(s_1, t_2)	$\cdots$	(s_1, t_n)
s_2	(s_2, t_1)	(s_2, t_2)	$\cdots$	(s_2, t_n)
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
s_m	(s_m, t_1)	(s_m, t_2)	$\cdots$	(s_m, t_n)

By definition the number of cells in this rectangle is mn. Hence

$$\#(S \times T) = \#(\text{cells in rectangle}) = mn = \#S \times \#T.$$

(b) To define a function $f : S \to T$ we need to specify an element $f(s) \in T$ for each element $s \in S$. Observe that for each $s \in S$ there are exactly $\#T$ possible choices for $f(s) \in T$. Thus the total number of choices is

$$\underbrace{\#T \times \#T \times \cdots \times \#T}_{\#S \text{ times}} = (\#T)^{(\#S)}.$$

(c) According to parts (a) and (b), the number of functions from $\{T, F\} \times \{T, F\}$ to $\{T, F\}$ is

$$\#\{T, F\}^{(\#(\{T,F\}\times\{T,F\}))} = \#\{T, F\}^{(\#\{T,F\}\times\#\{T,F\})} = 2^{(2\times 2)} = 2^4 = 16.$$

[Remark: How many of these 16 functions have we seen in this course?]

5. Counting Subsets.

(a) Explicitly write down all of the subsets of $\{1, 2, 3\}$.
(b) Explicitly write down all of the functions $\{1, 2, 3\} \to \{T, F\}$.
(c) If S is any set, let $\mathrm{Sub}(S)$ be the set of subsets of S and let $\mathrm{Fun}(S, \{T, F\})$ be the set of functions from S to $\{T, F\}$. Find a $1 : 1$ correspondence between these two sets:

$$\mathrm{Sub}(S) \leftrightarrow \mathrm{Fun}(S, \{T, F\}).$$

(d) If S is a finite set with $\#S$ elements, use Problem 4(b) to conclude that S has $2^{\#S}$ different subsets.

(a) Here they are. Note that there are 8 subsets. I wonder why.

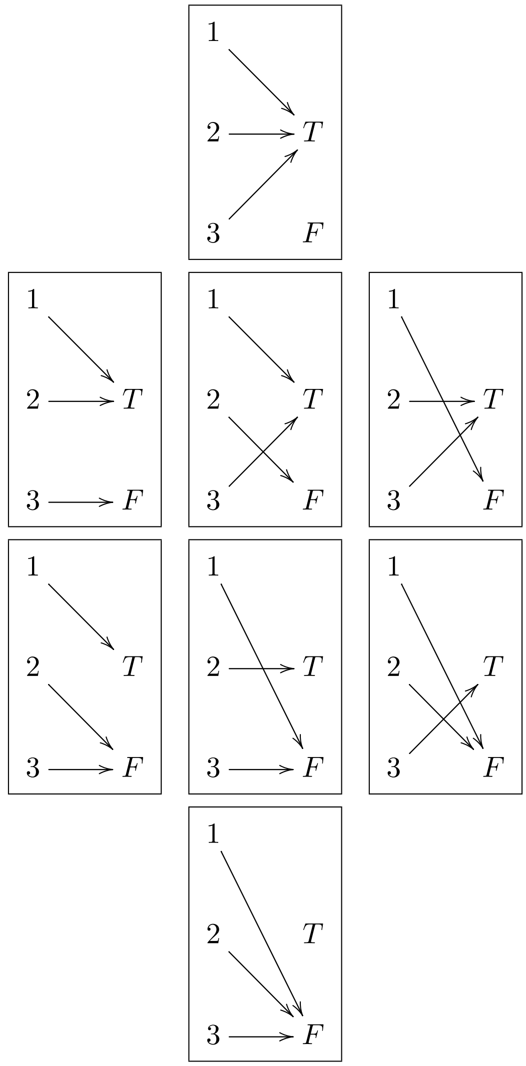
$$\{1, 2, 3\}$$

$$\{1, 2\} \quad \{1, 3\} \quad \{2, 3\}$$

$$\{1\} \quad \{2\} \quad \{3\}$$

$$\emptyset$$

(b) Here they are. There are $\#\{T, F\}^{\#\{1,2,3\}} = 2^3 = 8$ of them, as expected.



(c) For any subset $A \subseteq S$ we will define a function $f_A : S \to \{T, F\}$ as follows:

$$f_A(x) := \begin{cases} T & \text{if } x \in A, \\ F & \text{if } x \notin A. \end{cases}$$

For example, compare the pictures in parts (a) and (b). Note that $A \mapsto f_A$ defines a function from $\text{Sub}(S)$ to $\text{Fun}(S, \{T, F\})$. Is this function invertible? Yes. For any function $f : S \to \{T, F\}$ we define a set A by $\{x \in S : f(x) = T\}$ and observe that $f_A = f$.

(d) Since the function in part (c) is invertible (i.e., a bijection) we conclude that

$$\#\text{Sub}(S) = \#\text{Fun}(S, \{T, F\}),$$

and from Problem 4(b) we have

$$\#\text{Fun}(S, \{T, F\}) = \#\{T, F\}^{\#S} = 2^{\#S}.$$

Putting these equations together gives

$$\#\text{Sub}(S) = 2^{\#S}.$$