Math 309 | **Summer 2014**
Final Exam | **Drew Armstrong**

There are 5 problems, each with 3 parts. Each part is worth 2 points, for a total of 30 points. There is an optional bonus problem at the end. The value of the bonus problem is intangible.

1. Boolean Algebra.

(a) Draw the truth table for $P \Rightarrow Q$.

Here is the truth table. For fun, we also observe that $P \Rightarrow Q = (\neg P) \vee Q$.

P	Q	$\neg P$	$(\neg P) \vee Q$	$P \Rightarrow Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

(b) Prove that $(P \Rightarrow Q) = (\neg Q \Rightarrow \neg P)$ using a truth table.

Observe that the final columns in both tables are the same.

P	Q	$\neg Q$	$\neg P$	$\neg Q \Rightarrow \neg P$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

Alternatively, here's an abstract-algebraic proof:

$$(\neg Q \Rightarrow \neg P) = ((\neg\neg Q) \vee \neg P) = (Q \vee \neg P) = (\neg P \vee Q) = (P \Rightarrow Q).$$

(c) Express the statement $P \Leftrightarrow Q$ using only the boolean operations $\vee, \wedge, \neg$.

Recall that $P \Leftrightarrow Q$ means $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. Therefore from the observation in part (a) we can write

$$(P \Leftrightarrow Q) = (P \Rightarrow Q) \wedge (Q \Rightarrow P) = (\neg P \vee Q) \wedge (\neg Q \vee P).$$

Alternatively, we could first draw the truth table:

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Since this function has T's in the $P \wedge Q$ row and the $\neg P \wedge \neg Q$ row, the disjunctive normal form is

$$(P \Leftrightarrow Q) = (P \wedge Q) \vee (\neg P \wedge \neg Q).$$

2. Induction. Your goal in this problem is to prove the following identity by induction.

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1.$$

(a) State **exactly** what you want to prove. Make sure to define $P(n)$.

For all integers $n \geq 1$ we define the statement

$$P(n) = \text{“}1 \cdot 1! + 2 \cdot 2! + \cdots n \cdot n! = (n+1)! - 1.\text{”}$$

We will use induction to prove that $P(n)$ is true for all $n \geq 1$.

(b) State and prove the base case.

We observe that the statement $P(1)$ is true:

$$P(1) = \text{“}1 \cdot 1! = (1+1)! - 1\text{”} = \text{“}1 = 2 - 1\text{”} = T.$$

(c) State the prove the induction step.

Now consider an arbitrary integer $k \geq 1$ and let us assume for induction that $P(k)$ is true. In other words, let us assume that

$$1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k+1)! - 1.$$

But then we have

$$\begin{aligned}
&1 \cdot 1! + 2 \cdot 2! + \cdots + (k+1) \cdot (k+1)! \\
&= [1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k!] + (k+1) \cdot (k+1)! \\
&= [(k+1)! - 1] + +(k+1) \cdot (k+1)! && \text{induction} \\
&= [(k+1)! + (k+1) \cdot (k+1)!] - 1 \\
&= [1 + (k+1)] \cdot (k+1)! - 1 \\
&= (k+2) \cdot (k+1)! - 1 \\
&= (k+2)! - 1,
\end{aligned}$$

which means that $P(k+1)$ is also true.

[Remark: Where did I come up with this identity? Consider the collection of all words that can be made with the symbols $a_1, a_2, \ldots, a_{n+1}$. We will say the the symbol a_i is “happy” if it is placed in the ith position from the left. Note that every word except $a_1 a_2 \cdots a_{n+1}$ has at least one unhappy symbol. Therefore the number of words with at least one unhappy symbol is $(n+1)! - 1$. On the other hand, let us consider the collection of words in which the **leftmost unhappy symbol** occurs in the kth position from the **right**. One can argue that there are $(k-1) \cdot (k-1)!$ such words. Now sum over k.]

3. Binomial Theorem.

(a) Accurately state the Binomial Theorem.

Fix a non-negative integer $n \geq 0$. Then for all numbers x and y we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k y^{n-k}.$$

(b) Prove that a set with n elements has an equal number of "even subsets" (subsets with an even number of elements) and "odd subsets" (subsets with an odd number of elements). [Hint: Just plug something in.]

Since the binomial theorem is true for all numbers x and y, we may substitute $x = -1$ and $y = 1$ to obtain

$$\begin{aligned}
(-1+1)^n &= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + \binom{n}{n}(-1)^n \\
0 &= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + \binom{n}{n}(-1)^n \\
\binom{n}{1} + \binom{n}{3} + \cdots &= \binom{n}{0} + \binom{n}{2} + \cdots
\end{aligned}$$

Since $\binom{n}{k}$ is the number of subsets with size k, the last equation tells us that the number of odd-sized subsets equals the number of even-sized subsets.

(c) How many subsets of $\{1, 2, 3, 4, 5, 6\}$ have an **even** number of elements?

The total number of subsets of $\{1, 2, 3, 4, 5, 6\}$ is

$$2^{\#\{1,2,3,4,5,6\}} = 2^6 = 64.$$

Now let E and O be the numbers of even and odd subsets, so that $E + O = 64$. But we know from part (b) that $E = O$, so that

$$\begin{aligned}
E + O &= 64 \\
E + E &= 64 \\
2E &= 64 \\
E &= 32.
\end{aligned}$$

[Remark: In general, the number of even subsets of $\{1, 2 \ldots, n\}$ is 2^{n-1}.]

4. Probability. Consider a biased coin with $P(\text{"heads"}) = 1/3$.

(a) If you flip the coin n times. What is the probability that you get "heads" **exactly** k times?

The probability of getting heads exactly k times in n flips of a coin is

$$\binom{n}{k} P(\text{"heads"})^k P(\text{"tails"})^{n-k} = \binom{n}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{n-k} = \binom{n}{k} \frac{2^{n-k}}{3^n}$$

(b) If you flip the coin 5 times, what is the probability that you get "heads" an **even** number of times?

In this case we have $n = 5$. To compute the probability of an even number of heads, we sum the probabilities from (a) over all even values of k:

$$\begin{aligned}\binom{5}{0}\frac{2^{5-0}}{3^5} + \binom{5}{2}\frac{2^{5-2}}{3^5} + \binom{5}{4}\frac{2^{5-4}}{3^5} &= \binom{5}{0}\frac{32}{243} + \binom{5}{2}\frac{8}{243} + \binom{5}{4}\frac{2}{243} \\ &= 1 \cdot \frac{32}{243} + 10 \cdot \frac{8}{243} + 5 \cdot \frac{2}{243} \\ &= \frac{122}{243} = 50.2\%\end{aligned}$$

(c) If you flip the coin 111 times, how many times do you **expect** to get "heads"?

Consider a general coin with $P(\text{"heads"}) = p$ and $P(\text{"tails"}) = 1 - p$. If we flip this coin n times then on average we will expect to get heads pn times.

Since our coin has $p = 1/3$, if we flip the coin $n = 111$ times then on average we expect to see heads

$$np = 111 \cdot 1/3 = 37 \text{ times.}$$

5. Integers.

(a) Accurately state the Division Theorem for integers. [Hint: For all $a, b \in \mathbb{Z}$ with $b \neq 0$ ...]

For all integers $a, b \in \mathbb{Z}$ with $b \neq 0$, there exist unique integers $q, r \in \mathbb{Z}$ satisfying the following two properties:

$$\begin{cases} a = qb + r, \\ 0 \leq r < |b|. \end{cases}$$

(b) Accurately state the definition of an "even" integer.

We say an integer is even if it is "divisible by 2." In other words:

$$\text{“}n \text{ is even”} = \text{“}2|n\text{”} = \text{“}\exists k \in \mathbb{Z}, 2k = n.\text{”}$$

(c) Consider an integer $n \in \mathbb{Z}$. Prove that if n^2 is even then n is even.

We wish to prove that $2|n^2$ implies $2|n$. In order to do this we will instead prove the (equivalent) contrapositive statement that $2 \nmid n$ implies $2 \nmid n^2$. We will also use the fact (proved from the division theorem) that every non-even (i.e., odd) number has the form $2k + 1$ for some $k \in \mathbb{Z}$.

So let us suppose that $n \in \mathbb{Z}$ is odd; say $n = 2k + 1$ for some $k \in \mathbb{Z}$. It follows that

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2 \cdot (\text{some integer}) + 1$$

is also odd. □

6. Bonus. Give a **counting proof** of the following identity:

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Proof: Consider integers $0 \le k \le n$. From a bag of n unlabeled apples we will choose k apples to receive stickers. One of these k apples will receive **two** stickers and the other $k-1$ will receive **one** sticker each. We will count the possibilities in two ways.

On the one hand, we can choose the k stickered apples in $\binom{n}{k}$ ways. Then there are $k = \binom{k}{1}$ ways to choose the apple that will receive two stickers. This gives a total of

$$\binom{n}{k} \times k \quad \text{choices.}$$

On the other hand, we could first choose the two-stickered apple. There are $n = \binom{n}{1}$ ways to do this. Then we could choose $k-1$ apples from the remaining $n-1$ apples to receive one sticker each. There are $\binom{n-1}{k-1}$ ways to do this, for a total of

$$n \times \binom{n-1}{k-1} \quad \text{choices.}$$

Since these two formulas count the same things, they must be equal.

□