8/4/14

Quiz 5 now (20 minutes).
You may resubmit HW5 on Wednesday.

Plan for final week
Mon / Tues a bit of probability
Wed review
Thurs no class
lunch 1pm (?) Moon Thai & Japanese
Friday final exam

After discussing Quiz 5, here is a template for proof by induction.

Given $n \in \mathbb{N}$ define the statement

$P(n) :=$ "some statement depending on n".

We will prove by induction that $P(n)$ is true for all $n \geq 0$.

First we check the base case.

[If the base case is $n=0$, verify here that $P(0)$ is a true statement.]

Next we verify the induction step.

So, consider any $k \geq 0$ and assume for induction that $P(k)$ is true. In this case we want to show that $P(k+1)$ is also true.

[Now prove that $P(k+1)$ is true, using your assumption that $P(k)$ is true.]

We conclude by induction that

$$P(n) = T \text{ for all } n \geq 0$$

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Please use this template when rewriting your proofs for HW 5.

What shall we do for the final two lectures?

"A bit of probability."

Suppose we do an experiment where the outcome is a natural number.

[For Example: Flip a coin n times and record the number of heads.]

For all $k \in \mathbb{N}$ let p_k = the probability that we get k. We can record this information in a <u>generating function</u>

$$P(x) = \sum_{k \geq 0} p_k x^k$$

$$= p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots$$

Note that we must have

$$\begin{aligned} P(1) &= p_0 + p_1 \cdot 1 + p_2 \cdot 1^2 + p_3 \cdot 1^3 + \cdots \\ &= p_0 + p_1 + p_2 + p_3 + \cdots \\ &= \underline{1} \end{aligned}$$

because the total probability must sum to $\underline{1}$.

Example: Flip a fair coin 4 times and count the number of "heads".

Recall that we have

$$p_k = \begin{cases} \binom{4}{k}/2^4 & 0 \le k \le 4 \\ 0 & \text{otherwise} \end{cases}$$

so the generating function is

$$P(x) = \frac{1}{16} + \frac{4}{16}x + \frac{6}{16}x^2 + \frac{4}{16}x^3 + \frac{1}{16}x^4$$

Note that

$$P(1) = \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} + \frac{1}{16} = \frac{16}{16} = 1$$

as expected.

Q: Why bother with generating functions?

A: Because they make it easier to do computations.

For example, to compute the expected value of the experiment we just compute the derivative of the generating function and then substitute $x = 1$.

If $P(x) = \sum_{k \geq 0} p_k x^k$ then we have

$$P'(x) = \sum_{k \geq 0} k p_k x^{k-1}$$

and hence

$$P'(1) = \sum_{k \geq 0} k p_k \cdot 1^{k-1}$$

$$= \sum_{k \geq 0} k \cdot p_k = \text{the expected value} \checkmark.$$

In our example we have

$$P'(x) = 0 + \frac{4}{16} + \frac{6}{16} \cdot 2x + \frac{4}{16} \cdot 3x^2 + \frac{1}{16} 4 \cdot x^3$$

$$P'(1) = 0 + \frac{4}{16} + 2 \cdot \frac{6}{16} + 3 \cdot \frac{4}{16} + 4 \cdot \frac{1}{16}$$

$$= \frac{0 + 4 + 12 + 12 + 4}{16}$$

$$= \frac{32}{16} = 2.$$

Out of 4 flips of a fair coin we expect 2 heads. (No surprise.)

Now let's do the general case. Flip a biased coin with Prob("heads") = p exactly n times and count the number of heads you get.

We know from before that the probability of getting k heads is

$$p_k = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

So the generating function is

$$P(x) = \sum_k p_k x^k$$

$$= \sum_k \binom{n}{k} p^k (1-p)^{n-k} x^k$$

$$= \sum \binom{n}{k} (xp)^k (1-p)^{n-k}$$

$= ?$ Hmm, I recognize this.

$$= (xp + (1-p))^n$$

by the Binomial Theorem.

That's nice. Now let's compute the expected number of heads

$$P(x) = (xp+1-p)^n$$

$$P'(x) = n(xp+1-p)^{n-1} \cdot p$$

$$= pn(xp+1-p)^{n-1}$$

Substitute $x=1$ to get

$$P'(1) = pn(1\cancel{p}+1-\cancel{p})^{n-1}$$

$$= pn(1)^{n-1}$$

$$= pn.$$

Out of n coin flips we expect to get heads pn times.

OK, so this proof was simpler but we already knew this fact. Can we do new things with generating functions?

Yes.

Experiment: Consider a biased coin with Prob("heads") = p. Keep flipping it until you get "tails" and then stop. Report how many flips you did.

Let p_k = the probability that your first "tail" comes on the kth flip.

This means that you got "heads" $k-1$ times and then "tails" once, so

$$p_k = p^{k-1}(1-p).$$

Do you believe this? Let's just proceed and see what happens.

The generating function is

$$\begin{aligned} P(x) &= \sum_{k\geq 1} p_k x^k \\ &= \sum_{k\geq 1} p^{k-1}(1-p)x^k \\ &= x(1-p)\sum_{k\geq 1} p^{k-1}x^{k-1} \\ &= xp \sum_{k\geq 1} (xp)^{k-1} \end{aligned}$$

$$= x(1-p) \sum_{\ell \geq 0} (xp)^{\ell}$$

$$= x(1-p) \Big[\underbrace{1 + xp + (xp)^2 + (xp)^3 + \cdots}_{\text{this is a geometric series ☺}} \Big]$$

$$= x(1-p) \cdot \frac{1}{1-xp}$$

$$= \frac{x(1-p)}{1-xp}.$$

Hey, that's a nice formula. Let's check that $P(1) = 1$:

$$P(1) = \frac{1(1-p)}{1-1\cdot p} = \frac{1-p}{1-p} = 1 \quad \checkmark$$

This means we probably didn't make a mistake. Now we can use the "quotient rule" to compute the expected outcome.

$$p'(x) = \frac{(1-xp)(1-p) - x(1-p)(-p)}{(1-xp)^2}$$

$$= \frac{(1-p)}{(1-xp)^2}\left[1 - \cancel{x}p + \cancel{x}p\right]$$

$$p'(1) = \frac{(1-p)}{(1-1\cdot p)^2} = \frac{1}{1-p}.$$

What does this mean?

If $P(\text{"heads"}) = p$ then we expect to get "tails" for the first time on the

$\left(\frac{1}{1-p}\right)$th flip of the coin.

8/5/14

HW 5 rewrite due tomorrow.

Today: a bit more fun

Tomorrow: Review for final exam

Thurs: No Class (lunch 1pm)

Friday: Final exam

At the end of last class we proved the following.

Consider a biased coin with P("heads") = p. Flip the coin until you get "tails" then stop. Let p_k = the probability you got first "tails" on the kth flip.

We have $p_k = \underbrace{p^{k-1}}_{\text{"heads" } k-1 \text{ times}} \underbrace{(1-p)}_{\text{then "tails" once}}$

So the generating function is

$$P(x) = \sum_{k \geq 1} p^{k-1}(1-p)x^k$$

$$= \frac{x(1-p)}{1-xp}$$ after simplifying and using a geometric series.

This allows us to compute the expected number of flips to get "tails".

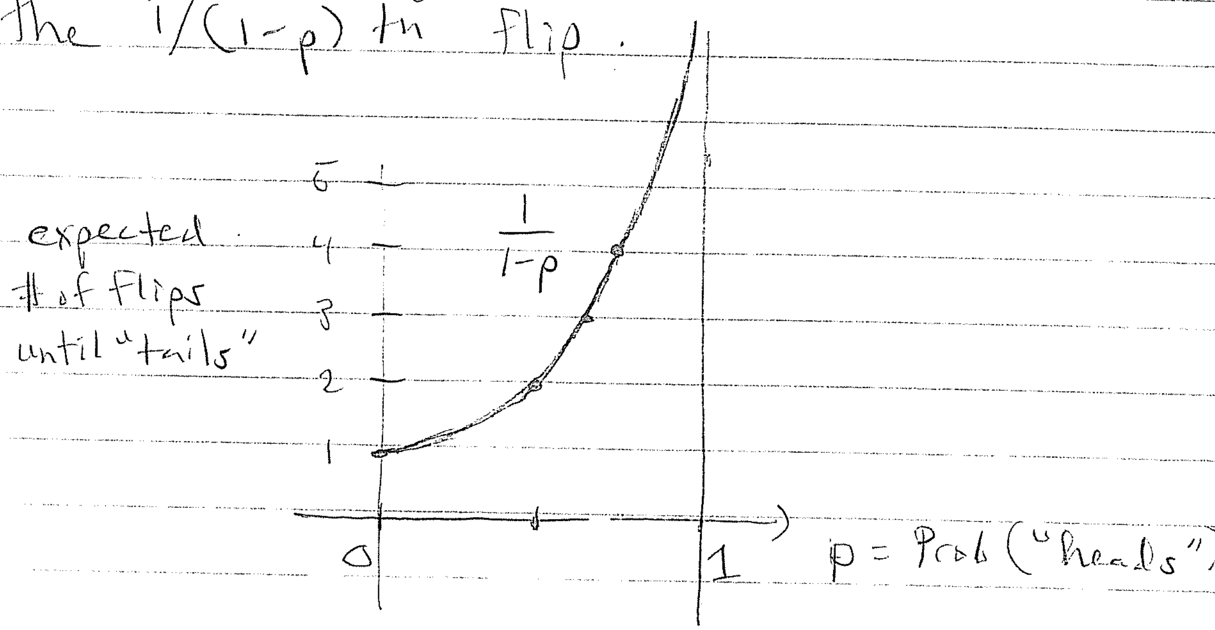
$$P'(x) = \frac{(1-xp)(1-p) - x(1-p)(-p)}{(1-xp)^2}$$

$$= \frac{(1-p)(1-\cancel{xp}+\cancel{xp})}{(1-xp)^2}$$

$$= \frac{(1-p)}{(1-xp)^2}$$

$$P'(1) = \frac{(1-p)}{(1-p)^2} = \frac{1}{1-p}.$$

We expect to get the first "tails" on the $1/(1-p)$ th flip.



We can use this to solve a fun problem.

The "Coupon collector" problem:

Suppose every box of Froot Loops contains one of 6 random toys. We keep buying Froot Loops until we get all 6 toys, then stop.

Q: How many boxes of Froot Loops do we expect to buy?

Actually, let's solve the general problem with n toys.

Instead of one experiment, we can think of this as a sequence of experiments

- Buy boxes until you get the first new toy then stop. (Obviously, we will get it in the first box.)

- Buy boxes until you get the second new toy then stop.

  How long will this take?

We can think of each box as a "coin flip" with

$$\text{Prob}(\text{"old toy"}) = 1/n$$
$$\text{Prob}(\text{"new toy"}) = (n-1)/n$$

From the previous discussion we expect to get a "new toy" after

$$\frac{1}{1-\frac{1}{n}} = \frac{1}{(n-1)/n} = \frac{n}{n-1} \quad \text{flips / boxes.}$$

- Buy boxes until you get the <u>third</u> new toy, then stop.

This is a "coin flip" with

$$\text{Prob}(\text{"old toy"}) = 2/n$$
$$\text{Prob}(\text{"new toy"}) = (n-2)/n.$$

So we expect to buy

$$\frac{1}{1-\frac{2}{n}} = \frac{1}{(n-2)/n} = \frac{n}{n-2} \quad \text{boxes.}$$

- To get the fourth new toy we expect to buy

$$\frac{n}{n-3} \text{ boxes.}$$

etc.

- After we have $n-1$ of the toys, we expect to buy

$$\frac{n}{n-(n-1)} = n \text{ boxes}$$

to get the final toy. DONE.

How many boxes did we buy in total?

$$\frac{n}{n-0} + \frac{n}{n-1} + \frac{n}{n-2} + \frac{n}{n-3} + \cdots + \frac{n}{n-(n-1)}$$

$$= \sum_{k=0}^{n-1} \frac{n}{n-k} = n \sum_{k=0}^{n-1} \frac{1}{n-k}$$

$$= n \sum_{l=1}^{n} \frac{1}{l}$$

$$= n\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right).$$

That's the answer.

So, if we want to collect 6 toys, we expect to buy

$$6\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right)$$

$$= \cancel{6}^{3} \cdot \frac{49}{\cancel{20}_{10}} = \frac{147}{100} = 14.7$$

boxes of Froot Loops.

Thinking Problem:

We define the "harmonic number"

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n}.$$

How does H_n grow asymptotically?

$$H_n \sim ?$$

To end the course I will mention a major theorem.

Flip a biased coin n times. The probability of getting k heads is

$$\binom{n}{k} p^k (1-p)^{n-k}$$

But for large n this may be impossible to compute. In practice we must use a numerical approximation called the

☆ Central Limit Theorem:

For n large and k close to np we have

$$\binom{n}{k} p^k (1-p)^{n-k} \approx \frac{1}{\sqrt{2\pi np(1-p)}} e^{\frac{-(k-np)^2}{2np(1-p)}}$$

This is only approximate, but it is <u>much</u> <u>easier</u> <u>to</u> <u>compute</u>.

This theorem was first discovered by Abraham de Moivre in 1738 for the case of a fair coin.

We can prove de Moivre's theorem without too much trouble if you are willing to assume

☆ Stirling's Approximation:

For large n we have

$$\boxed{n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n}.$$

We also need the Taylor series for the natural logarithm. Recall that for small x we have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

Substitute $x \to -x$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

Take the antiderivative of both sides

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

For x <u>very</u> small we have

$$\boxed{\ln(1+x) \approx x}$$

Finally, we need one more ingredient.
For large n we have

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} \approx \frac{\sqrt{4\pi n}\,(2n/e)^{2n}}{\left(\sqrt{2\pi n}\,(n/e)^n\right)^2}$$

$$= \frac{\sqrt{4\pi n}\,(2n)^{2n}\,\cancel{e^{-2n}}}{2\pi n\; n^{2n}\,\cancel{e^{-2n}}} = \frac{\cancel{2}\sqrt{\pi n}}{\cancel{2}\pi n}\left(\frac{2\cancel{n}}{\cancel{n}}\right)^{2n}$$

$$= \frac{1}{\sqrt{\pi n}} \cdot 2^{2n} = \frac{1}{\sqrt{\pi n}}\,4^n .$$

So,

$$\boxed{\binom{2n}{n} \approx \frac{4^n}{\sqrt{\pi n}}}$$

Without further ado, here is the proof of de Moivre's "central limit theorem".

Proof: For n large and k small we will approximate the quantity

$$X = \binom{2n}{n+k} \Big/ \binom{2n}{n}.$$

$$= \frac{\cancel{(2n)}!}{(n+k)!\,(n-k)!} \cdot \frac{n!\,n!}{\cancel{(2n)}!}$$

$$= \frac{n!}{(n+k)!} \cdot \frac{n!}{(n-k)!}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)}{(n+k)(n+k-1)\cdots(n+1)}$$

(Divide top and bottom by n^k)

$$= \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)}{\left(1+\frac{k}{n}\right)\left(1+\frac{k-1}{n}\right)\cdots\left(1+\frac{1}{n}\right)}.$$

Now take the natural log of X and use the approximation $\ln(1+x) \approx x$ for small x.

$$\ln(X) = \ln\left(\frac{(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k-1}{n})}{(1+\frac{k}{n})(1+\frac{k-1}{n})\cdots(1+\frac{1}{n})}\right)$$

$$= \ln\left(1-\frac{1}{n}\right) + \ln\left(1-\frac{1}{2}\right) + \cdots + \ln\left(1-\frac{k-1}{n}\right)$$

$$-\ln\left(1+\frac{k}{n}\right) - \ln\left(1+\frac{k-1}{n}\right) - \cdots - \ln\left(1+\frac{1}{n}\right)$$

$$\approx -\frac{1}{n} - \frac{2}{n} - \cdots - \frac{k-1}{n}$$

$$-\frac{1}{n} - \frac{2}{n} - \cdots - \frac{k-1}{n} - \frac{k}{n}$$

$$= -\frac{2}{n}\left(1+2+3+\cdots+(k-1)\right) - \frac{k}{n}$$

$$= -\frac{\cancel{2}}{n}\,\frac{k(k-1)}{\cancel{2}} - \frac{k}{n}$$

$$= -\frac{k(k-1)}{n} - \frac{k}{n}$$

$$= \frac{-k^2 + \cancel{k} - \cancel{k}}{n} = -k^2/n$$

We have shown that for k/n small we have

$$\ln\left(\frac{\binom{2n}{n+k}}{\binom{2n}{n}}\right) \approx -k^2/n$$

$$\frac{\binom{2n}{n+k}}{\binom{2n}{n}} \approx e^{-k^2/n}$$

$$\binom{2n}{n+k} \approx \binom{2n}{n} e^{-k^2/n}$$

and for n large we have

$$\binom{2n}{n+k} \approx \frac{4^n}{\sqrt{\pi n}} e^{-k^2/n}$$

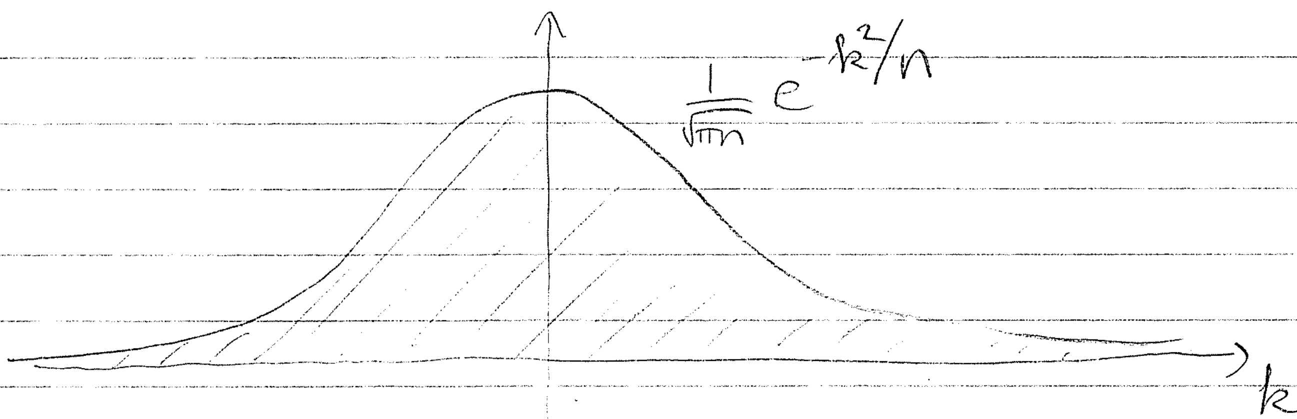
Great, but what does this have to do with probability?

Flip a fair coin $2n$ times for n large.
Then if k/n is small the probability
of getting $n+k$ "heads" is

$$\frac{\binom{2n}{n+k}}{(2)^{2n}} \approx \frac{\frac{\cancel{4^n}}{\sqrt{\pi n}} e^{-k^2/n}}{\cancel{(2)^{2n}}}$$

$$= \frac{1}{\sqrt{\pi n}} e^{-k^2/n}$$

This is beautiful. For large n we
can replace coin flipping with a
"normal distribution"



Total area is 1 (as it should be)

Example: Flip a fair coin 3600 times. What is the probability we get between 1770 and 1830 "heads"?

By de Moivre's approximation, the probability is

$$\sum_{k=-30}^{30} \frac{\binom{3600}{1800+k}}{4^{1800}} \approx \int_{-30}^{30} \frac{1}{\sqrt{\pi \cdot 1800}} e^{-k^2/1800} \, dk$$

hard easy

My computer evaluates the integral and tells me that the probability is

$\approx 68.2688\,\%$

Thank you de Moivre :)