Math 309 **Summer 2014**
Quiz 5 Solutions **Drew Armstrong**

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1. Accurately state the the Principle of Induction for $\mathbb{N}$.

Consider any function $P : \mathbb{N} \to \{T, F\}$. If

- $P(b) = T$ for some $b \in \mathbb{N}$, and
- $P(k) \Rightarrow P(k+1)$ for all $k \geq b$,

then we have $P(n) = T$ for all $n \geq b$.

2. Consider the sequence defined recursively by $a_0 = 1$, $a_1 = 3$, and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 2$. Compute a_5.

We record the sequence in a table:

n	0	1	2	3	4	5
a_n	1	3	4	7	11	18

Note that $a_5 = 18$.

3. For the sequence in Problem 2, what value does a_n/a_{n-1} approach as $n \to \infty$?

If we divide the equation $a_n = a_{n-1} + a_{n-2}$ by a_{n-1} we get

$$\frac{a_n}{a_{n-1}} = 1 + \frac{a_{n-2}}{a_{n-1}} = 1 + \frac{1}{a_{n-1}/a_{n-2}}.$$

If a_n/a_{n-1} approaches ℓ as $n \to \infty$, then so does a_{n-1}/a_{n-2} and the equation says

$$\ell = 1 + 1/\ell.$$

The solutions are $\ell = (1 \pm \sqrt{5})/2$. Since ℓ must be positive we conclude that $\ell = (1 + \sqrt{5})/2$.

Now consider the sequence defined recursively by $r_0 = 1$ and $r_n = 2 \cdot r_{n-1}$ for all $n \geq 1$. Let $P(n)$ be the statement " $r_n = 2^n$ ". You will prove by induction that $P(n) = T$ for all $n \geq 0$.

4. State and prove the base case.

The base case is $n = 0$. Note that the statement $P(0)$ is true because $1 = 2^0$.

5. State and prove the induction step.

Consider any $k \geq 0$ and **assume for induction** that $P(k) = T$. That is, assume that $r_k = 2^k$. In this case we want to show that $P(k+1) = T$. That is, we want to show that $r_{k+1} = 2^{k+1}$. Finally, observe that the recurrence and our assumption together imply that

$$r_{k+1} = 2 \cdot r_k = 2 \cdot 2^k = 2^{k+1},$$

as desired.