Math 309 **Summer 2014**
Homework 4 Solutions **Drew Armstrong**

Recall that we define the logical symbol "$\Rightarrow$" by "$P \Rightarrow Q$" := "$\neg P \vee Q$". We pronounce the statement "$P \Rightarrow Q$" as "if P then Q", or "P implies Q". Recall that we say an integer $n \in \mathbb{Z}$ is **even** if there exists $k \in \mathbb{Z}$ such tha $n = 2k$, and we say that $n \in \mathbb{Z}$ is **odd** if there exists $k \in \mathbb{Z}$ such that $n = 2k + 1$. On Problems 1–4 you will give a (hopefully) nice proof that for all $m, n \in \mathbb{Z}$ we have

$$(mn \text{ is even}) \Longleftrightarrow (m \text{ is even or } n \text{ is even}).$$

1. Use the contrapositive to show that "$P \Rightarrow (Q \vee R)$" = "$(\neg Q \wedge \neg R) \Rightarrow \neg P$".

Proof. Applying the contrapositive and then de Morgan's identity gives

$$\begin{aligned} \text{“}P \Rightarrow (Q \vee R)\text{”} &= \text{“}\neg(Q \vee R) \Rightarrow \neg P\text{”} \\ &= \text{“}(\neg Q \wedge \neg R) \Rightarrow \neg P\text{”}. \end{aligned}$$

In words, this means that the statement "if P is true then Q or R is true" is logically equivalent to the statement "if Q and R are both false then P is false". □

2. Use the result from Problem 1 to prove that if mn is even, then m is even or n is even. [Hint: Let $P =$"mn is even", $Q =$"m is even", and $R =$"n is even".]

Proof. If we let $P =$"mn is even", $Q =$"m is even", and $R =$"n is even", then the statement we want to prove is $P \Rightarrow (Q \vee R)$. By Problem 1 this is logically equivalent to the statement $(\neg Q \wedge \neg R) \Rightarrow \neg P$. In other words, "if m and n are both odd, then mn is odd". We will prove this statement.

So assume that m and n are both odd. This means that there exist integers $k, \ell \in \mathbb{Z}$ such that $m = 2k + 1$ and $n = 2\ell + 1$. Multiplying these gives

$$\begin{aligned} mn &= (2k + 1)(2\ell + 1) \\ &= 4k\ell + 2k + 2\ell + 1 \\ &= 2(2k\ell + k + \ell) + 1. \end{aligned}$$

Since $mn = 2(\text{some integer}) + 1$, we conclude that mn is odd as desired. □

3. Use a truth table to show that "$(P \vee Q) \Rightarrow R$" = "$(P \Rightarrow R) \wedge (Q \Rightarrow R)$".

Proof. Here is the truth table.

P	Q	R	$P \vee Q$	$(P \vee Q) \Rightarrow R$	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Note that the 5th and 8th columns are logically equivalent. In words, the statement "if P or Q is true then R is true" is logically equivalent to the statment "if P is true then R is true, **and** if Q is true then R is true". This doesn't sound as nice but it is an easier statement to prove, as we will see. $\square$

4. Use the result from Problem 3 to prove that if m is even or n is even, then mn is even. [Hint: What are P, Q, and R in this case?]

Proof. Let $P =$"m is even", $Q =$"n is even", and $R =$"mn is even. The statement we want to prove is $(P \vee Q) \Rightarrow R$. By Problem 3, this is equivalent to the statement $(P \Rightarrow R) \wedge (Q \Rightarrow R)$. In other words, "if m is even then mn is even, **and** if n is even then mn is even". These are two statements that we can prove separately.

First, assume that m is even, i.e., there exists $k \in \mathbb{Z}$ such that $m = 2k$. In this case we have $mn = 2kn = 2(kn) = 2(\text{some integer})$, and so mn is even as desired.

Sedond, assume that n is even, i.e., there exists $\ell \in \mathbb{Z}$ such that $n = 2\ell$. In this case we have $mn = m2\ell = 2(m\ell) = 2(\text{some integer})$, and so mn is even as desired. $\square$

5. Fill in the rest of this table. What does it remind you of? Use the table to express the statement "$m + n$ is even" in terms of the statements "m is even" and "n is even", together with the logical symbols $\vee, \wedge, \neg$.

m	n	mn	$m+n$
E	E	E	E
E	O	E	O
O	E	E	O
O	O	O	E

Proof. The symbols E and O represent "even" and "odd". The table reminds me of this:

P	Q	$P \vee Q$	$P \Leftrightarrow Q$
T	T	T	T
T	F	T	F
F	T	T	F
F	F	F	T

We can interpret $P =$"m is even" and $Q =$"n is even". Then the 3rd columns of the tables tell us that "mn is even" is logically equivalent to "m is even or n is even", which is what we proved in Problems 1–4.

The 4th columns of the tables tell us that "$m + n$ is even" is logically equivalent to "m is even" $\Leftrightarrow$ "n is even". Can we write this in terms of the symbols $\vee, \wedge, \neg$? Well, the disjunctive normal form of $P \Leftrightarrow Q$ is

$$\text{“}P \Leftrightarrow Q\text{”} = \text{“}(P \wedge Q) \vee (\neg P \wedge \neg Q)\text{”}.$$

So we can say that "$m + n$ is even" is logically equivalent to "m and n are both even, or m and n are both odd". This is not surprising if you think about it. $\square$

[Remark: We have just discovered that the properties of being even and being odd form some kind of algebra with two elements called E and O. They can be added and multiplied.]