Math 309 — Summer 2014
Homework 1 Solutions — Drew Armstrong

Let L_n be the maximum number of regions we can get by drawing n (infinite) lines in the plane. We showed in class that

$$L_n = 1 + (1 + 2 + 3 + \cdots + n) = 1 + \sum_{k=1}^{n} k = 1 + \frac{n(n+1)}{2} = \frac{n^2 + n + 2}{2}.$$

1. Let f and g be two functions of a discrete variable n. We write $f(n) \sim g(n)$ (and we say that $f(n)$ is asymptotic to $g(n)$) if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

Show that $L_n \sim \frac{1}{2}n^2$.

Proof. Let $f(n) = L_n = (n^2 + n + 2)/2$ and $g(n) = n^2/2$. We want to show that $f(n) \sim g(n)$. By definition this means we must show that $\lim_{n\to\infty} f(n)/g(n) = 1$. First we examine the function:

$$\frac{f(n)}{g(n)} = \frac{(n^2 + n + 2)/2}{n^2/2} = \frac{n^2 + n + 2}{n^2} = 1 + \frac{1}{n} + \frac{2}{n^2}.$$

Now we can compute the limit:

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \left(1 + \frac{1}{n} + \frac{2}{n^2}\right) \\
&= \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} \frac{2}{n^2} \\
&= 1 + 0 + 0 \\
&= 1.
\end{aligned}$$

□

2. We proved in class that

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Use this formula to evaluate the sum $\sum_{k=1}^{n}(a + bk + ck^2)$, where a, b, c are arbitrary constants.

Proof. By rearranging the order of the sum and factoring out constant multiples we have

$$\begin{aligned}
\sum_{k=1}^{n}(a+bk+ck^2) &= a\sum_{k=1}^{n}1+b\sum_{k=1}^{n}k+c\sum_{k=1}^{n}k^2 \\
&= an+b\frac{n(n+1)}{2}+c\frac{n(n+1)(2n+1)}{6} \\
&= \frac{n}{6}\left(6a+3b(n+1)+c(n+1)(2n+1)\right) \\
&= \frac{n}{6}\left(6a+3bn+3b+c(2n^2+3n+1)\right) \\
&= \frac{n}{6}\left((2c)n^2+3(b+c)n+(6a+3b+c)\right) \\
&= \left(\frac{c}{3}\right)n^3+\left(\frac{b+c}{2}\right)n^2+\left(\frac{6a+3b+c}{6}\right)n.
\end{aligned}$$

□

3. Now let P_n be the maximum number of 3-dimensional regions we can get by cutting 3-dimensional space with n infinite planes (i.e., the maximum number of pieces of cheese we can get using n cuts). You can assume that for all $n > 0$ we have

$$P_{n+1} = P_n + L_n.$$

(a) Use this recurrence to show that for all $n > 0$ we have

$$P_n = 1 + L_0 + L_1 + L_2 + \cdots L_{n-1} = 1 + \sum_{k=0}^{n-1} L_k.$$

(b) Use the result of part (a) to show that for all $n > 0$ we have

$$P_n = \frac{n^3+5n+6}{6}.$$

[Hint: Use Problem 2.] It just so happens that this formula also works when $n = 0$.

Proof. First I'll mention the recurrence $P_{n+1} = P_n + L_n$. Suppose we have n planes dividing 3D space into P_n 3D regions. Now we add an $(n+1)$-st plane. If we do this correctly (no multiple intersections and no parallel planes) then the first n planes will intersect our new plane in n lines and divide the new plane into L_n regions. This means the new plane passes through exactly L_n of the P_n 3D regions and cuts each of these in two. This creates L_n **new** 3D regions, so the total is now $P_n + L_n$. On the other hand, the number of regions created by $n+1$ planes (if done correctly) is P_{n+1}. Hence $P_{n+1} = P_n + L_n$.

For part (a), first note that $P_0 = 1$. Now we use the recurrence to expand:

$$\begin{aligned}
&P_0 = 1 \\
&P_1 = P_0 + L_0 = 1 + L_0 \\
&P_2 = P_1 + L_1 = (1 + L_0) + L_1 \\
&P_3 = P_2 + L_2 = (1 + L_0 + L_1) + L_2 \\
&\quad\vdots \\
&P_n = 1 + L_0 + L_1 + L_2 + \cdots + L_{n-1} = 1 + \sum_{k=0}^{n-1} L_k.
\end{aligned}$$

Now we can use the formula $L_k = (k^2 + k + 2)/2 = k^2/2 + k/2 + 1$ and the technique from Problem 2 to find a closed formula:

$$\begin{aligned}
P_n &= 1 + \sum_{k=0}^{n-1} \left(\frac{1}{2}k^2 + \frac{1}{2}k + 1 \right) \\
&= 1 + \frac{1}{2}\sum_{k=0}^{n-1} k^2 + \frac{1}{2}\sum_{k=0}^{n-1} k + \sum_{k=0}^{n-1} 1 \\
&= 1 + \frac{1}{2}\sum_{k=1}^{n-1} k^2 + \frac{1}{2}\sum_{k=1}^{n-1} k + \sum_{k=0}^{n-1} 1 \\
&= 1 + \frac{1}{2}\frac{(n-1)((n-1)+1)(2(n-1)+1)}{6} + \frac{1}{2}\frac{(n-1)((n-1)+1)}{2} + n \\
&= 1 + \frac{1}{2}\frac{n(n-1)(2n-1)}{6} + \frac{1}{2}\frac{n(n-1)}{2} + n \\
&= n + 1 + \frac{n(n-1)}{2}\left(\frac{2n-1}{6} + \frac{1}{2} \right) \\
&= n + 1 + \frac{n(n-1)}{2}\left(\frac{n+1}{3} \right) \\
&= \frac{1}{6}\left(6(n+1) + n(n-1)(n+1)\right) \\
&= \frac{1}{6}\left(6n + 6 + n^3 - n\right) \\
&= \frac{n^3 + 5n + 6}{6}.
\end{aligned}$$

□

Jacob Steiner stopped with 3-dimensional cheese because in 1826 no one believed in 4-dimensional cheese. Today we do believe in 4-dimensional cheese (at least I do). Let $f_d(n)$ be the maximum number of pieces obtained when we make n $(d-1)$-dimensional cuts of a d-dimensional cheese. The same geometric argument "should" work to prove

$$f_d(n+1) = f_d(n) + f_{d-1}(n).$$

One can then use the recurrence to obtain the following nice formula:

$$f_d(n) = 1 + n + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{d}.$$

This formula was first written down by Ludwig Schläfli in the 1840s. We may return to it when we study binomial coefficients.