Math 309 **Fall 2022**
Homework 5 **Drew Armstrong**

1. Vertex Degrees.

(a) Explain why $1, 2, 3, 4, 5, 6$ cannot be the vertex degrees of a graph. [Hint: Handshake.]
(b) Explain why $1, 1, 1, 1, 1, 3$ cannot be the vertex degrees of a connected graph. [Hint: There are $n = 6$ vertices. Use Handshaking to find the number of edges e. A connected graph must have $n - 1 \leq e$.]
(c) Draw a non-connected graph with vertex degrees $1, 1, 1, 1, 1, 3$.

(a): Here are two solutions.

First, if we assume that the graph is simple (no loops, no multiple edges) then since there are 6 vertices, the maximum possible vertex degree is 5, since a vertex of degree 5 would be connected to every other vertex. The second solution applies to any kind of graph.

Second, the Handshaking Lemma applies to any graph:[1]

$$2 \cdot (\# \text{ of edges}) = \text{the sum of vertex degrees.}$$

A graph with vertex degrees $1, 2, 3, 4, 5, 6$ would have

$$2 \cdot (\# \text{ of edges}) = 1 + 2 + 3 + 4 + 5 + 6 = 21,$$

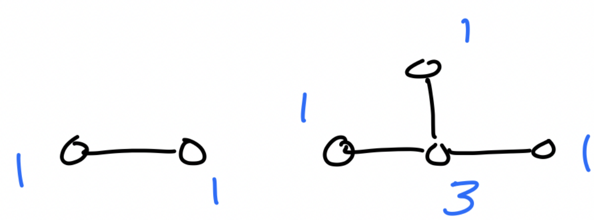
which is impossible.

(b): Any graph with vertex degrees $1, 1, 1, 1, 1, 3$ must have $n = 6$ vertices and

$$e = (1 + 1 + 1 + 1 + 1 + 3)/2 = 4$$

edges. But a connected graph must have $n - 1 \leq e$. Hence this graph cannot be connected.

(c): Here is a disconnected graph with vertex degrees $1, 1, 1, 1, 1, 3$:



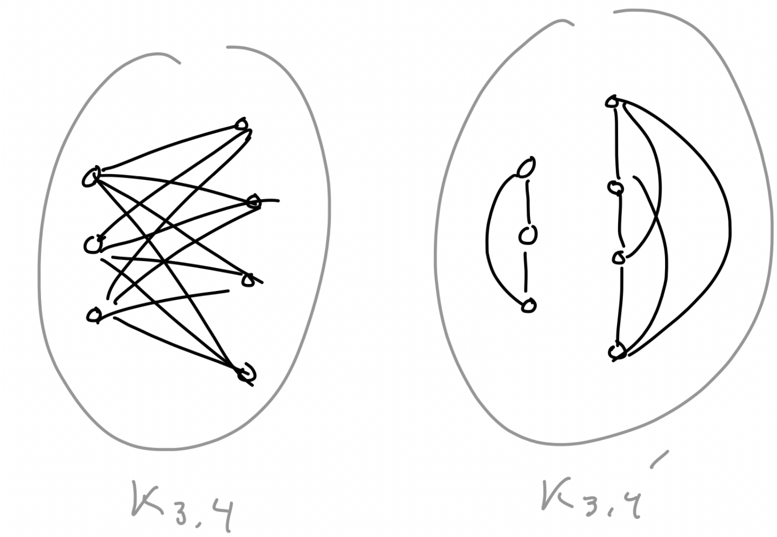
2. Complete Bipartite Graphs. The complete bipartite graph $K_{m,n}$ consists of $m + n$ vertices $\{u_1, \ldots, u_m, v_1, \ldots, v_n\}$ and mn edges $\{u_i, v_j\}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

(a) For a given graph G, let G' denote the complementary graph with edges and non-edges switched. Draw pictures of $K_{3,4}$ and its complement $K'_{3,4}$.
(b) For all m and n, explain why

$$(\# \text{ edges in } K_{m,n}) + (\# \text{ edges in } K'_{m,n}) = \binom{m+n}{2}.$$

(a): Here are pictures of $K_{3,4}$ and $K'_{3,4}$:

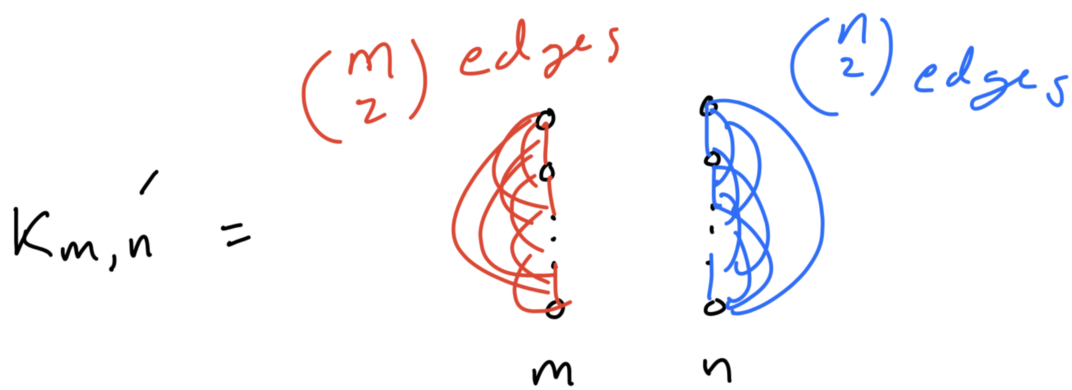
[1]This formula allows loops and multiple edges, assuming that a loop adds 2 to the degree of its vertex.



Note that $K_{3,4}$ has $3 \cdot 4 = 12$ edges and $K'_{3,4}$ has 9 edges.

(b): Here are two solutions.

In general, we note that $K'_{m,n}$ is a disjoint union of K_m and K_n:



Since K_m has $\binom{m}{2}$ edges (any two vertices are connected by an edge) we conclude that

$$
\begin{aligned}
(\#\text{ edges in } K_{m,n}) + (\#\text{ edges in } K'_{m,n}) &= mn + \binom{m}{2} + \binom{n}{2} \\
&= mn + \frac{m(m-1)(m-1)!}{2 \cdot (m-2)!} + \frac{n(n-1)(n-2)!}{2 \cdot (n-2)!} \\
&= mn + \frac{m(m-1)}{2} + \frac{n(n-1)}{2} \\
&= \frac{2mn + m(m-1) + n(n-1)}{2} \\
&= \frac{mn + m^2 - m + n^2 - n}{2}.
\end{aligned}
$$

On the other hand, we have

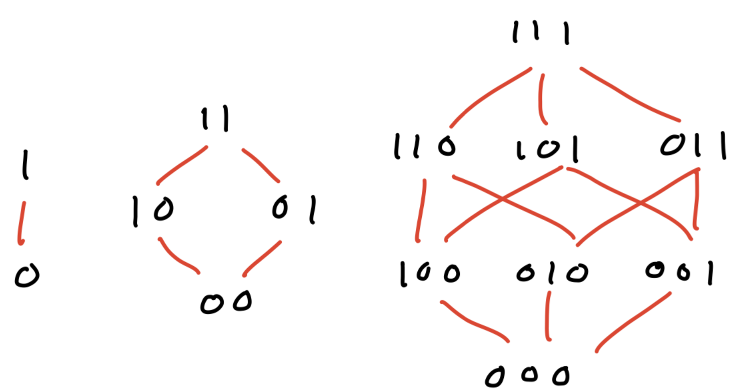
$$\begin{aligned}\binom{m+n}{2} &= \frac{(m+n)(m+n-1)(m+n-2)!}{2\cdot(m+n-2)!} \\ &= \frac{(m+n)(m+n-1)}{2} \\ &= \frac{m^2+mn-m+mn+n^2-n}{2} \\ &= \frac{2mn+m^2-m+n^2-n}{2}.\end{aligned}$$

Alternatively, we can give a counting proof. The graph $K_{m,n}$ has $m+n$ vertices, and there are $\binom{m+n}{2}$ possible edges among these vertices. Some of these edges occur in $K_{m,n}$ and the rest occur in the complement $K'_{m,n}$. That's it.

3. The Hypercube Graph. The hypercube graph Q_n has 2^n vertices, corresponding to the binary strings (words from the alphabet $\{0,1\}$) of length n. We draw an edge between two vertices if the corresponding words differ in a single position.

(a) Draw the graphs Q_1, Q_2 and Q_3.
(b) Compute the number of edges in Q_n. [Hint: Use the Handshaking Lemma. Note that each vertex of Q_n has the same degree.]

(a): Here are the graphs Q_1, Q_2 and Q_3, where I have labeled the vertices by their corresponding binary strings:



(b): Every vertex in Q_n corresponds to a binary string of length n. Let B be a binary string:

$$B = (b_1, b_2, \ldots, b_n) \quad \text{with } b_1,\ldots,b_n \in \{0,1\}.$$

There are exactly n strings that differ from B in a single position, since each of the n bits can be flipped. Hence every vertex in Q_n has degree n. Since Q_n has 2^n vertices, each with degree n, it follows from Handshaking that

$$2\cdot(\#\text{ edges in } Q_n) = \underbrace{n+n+\cdots+n}_{2^n \text{ times}} = n\cdot 2^n,$$

and hence

$$(\#\text{ edges in } Q_n) = \frac{n\cdot 2^n}{2} = n\cdot 2^{n-1}.$$

4. Tree Degrees. Let T be a tree with n vertices, and let n_k be the number of vertices of degree k, so that $n = \sum_k n_k$.

(a) Explain why $\sum_k k \cdot n_k = 2(n-1)$. [Hint: A tree with n vertices has $n-1$ edges.]

(b) Use part (a) to prove that

$$n_1 = 2 + n_3 + 2n_4 + 3n_5 + 4n_6 + \cdots .$$

(c) An *alkane* is a saturated hydrocarbon molecule. We can think of this as a tree with only vertices of degree 1 (hydrogen atoms) and degree 4 (carbon atoms). In this case, use part (b) to show that

$$(\text{\# of hydrogen atoms}) = 2 + 2(\text{\# of carbon atoms}).$$

(a): Let e be the number of edges in T. Since there are k vertices with degree n_k, the Handshaking Lemma gives

$$\begin{aligned} 2e &= \text{sum of vertex degrees} \\ &= \underbrace{1+1+\cdots+1}_{n_1 \text{ times}} + \underbrace{2+2+\cdots+2}_{n_2 \text{ times}} + \underbrace{3+3+\cdots+3}_{n_3 \text{ times}} + \cdots \\ &= 1 \cdot n_1 + 2 \cdot n_2 + 3 \cdot n_3 + \cdots . \end{aligned}$$

The sum stops after a finite number of steps because T has finitely many vertices.

(b): We also know that $e = n - 1$,[2] which is a property of trees, and $n = n_1 + n_2 + \cdots$, since n_k is the number of vertices of degree k, and n is the total number of vertices. Combining these with part (a) gives

$$\begin{aligned} 2e &= n_1 + 2 \cdot n_2 + 3 \cdot n_3 + \cdots \\ 2(n-1) &= n_1 + 2 \cdot n_2 + 3 \cdot n_3 + \cdots \\ 2n &= 2 + n_1 + 2 \cdot n_2 + 3 \cdot n_3 + \cdots \\ 2(n_1 + n_2 + n_3 + \cdots) &= 2 + n_1 + 2 \cdot n_2 + 3 \cdot n_3 + \cdots \\ 2 \cdot n_1 &= 2 + n_1 + 0 \cdot n_2 + 1 \cdot n_3 + 2 \cdot n_4 + \cdots \\ n_1 &= 2 + n_3 + 2 \cdot n_4 + 3 \cdot n_5 + \cdots . \end{aligned}$$

(c): A hydrocarbon molecule can be thought of as a graph with vertices of degree 1 (called hydrogen atoms) and vertices of degree 4 (called carbon atoms). The molecule is called "saturated" if the number of hydrogen atoms is as large as possible for the given number of carbon atoms. It turns out that this implies no cycles and no multiple edges (double and triple bonds), so the molecule must be a tree. In the formula from part (b) we have $n_k = 0$ for all k except for 1 and 4, hence

$$\begin{aligned} n_1 &= 2 + n_3 + 2 \cdot n_4 + 3 \cdot n_5 + \cdots \\ n_1 &= 2 + 0 + 2 \cdot n_4 + 3 \cdot 0 + \cdots \\ n_1 &= 2 + 2 \cdot n_4 \\ (\text{\# of hydrogen atoms}) &= 2 + 2(\text{\# of carbon atoms}). \end{aligned}$$

5. Planarity of Bipartite Graphs. Let G be a simple, bipartite graph (i.e., with no loops, no multiple edges, and no cycles of odd length) with v vertices and e edges.

[2] Recall that any graph with k connected components satisfies $n - k \le e$, so that any connected graph satisfies $n - 1 \le e$. A tree is a connected graph with the minimum possible number of edges.

(a) Suppose that G has a planar drawing with f faces. In this case, show that

$$2e \geq 4f.$$

[Hint: By the Handshaking Lemma, the sum of the degrees of the faces equals $2e$. By our assumptions on G, each face in the drawing must have degree ≥ 4.]

(b) Combine (a) with Euler's Formula $v - e + f = 2$ to show that

$$e \leq 2v - 4.$$

(c) Use part (b) to prove that the complete bipartite graph $K_{3,3}$ has no planar drawing.

(a): The Handshaking Lemma for planar graphs says that

$$2e = \sum \text{face degrees}.$$

A graph without loops has no faces of degree 1 and a graph without multiple edges has no faces of degree 2. Furthermore, a bipartite graph has no faces of odd degree, so every face in a planar drawing of a simple, bipartite graph must have degree ≥ 4. It follows that

$$\begin{aligned} 2e &= \sum \text{face degrees} \\ &\geq \underbrace{4 + 4 + \cdots + 4}_{f \text{ times}} \\ &= 4f. \end{aligned}$$

(b): Combining part (a) with Euler's formula $v - e + f = 2$ gives

$$\begin{aligned} 4f &\leq 2e \\ 2f &\leq e \\ 2(2 - v + e) &\leq e \\ 4 - 2v + 2e &\leq e \\ e &\leq 2v - 4. \end{aligned}$$

In summary, any simple, bipartite graph that has a planar drawing must satisfy $e \leq 2v - 4$.

(c): The complete bipartite graph $K_{3,3}$ has $v = 3 + 3 = 6$ vertices and $e = 3 \cdot 3 = 9$ edges, so that the inequality $e \leq 2v - 4$ is false. Hence $K_{3,3}$ cannot possibly have a planar drawing.

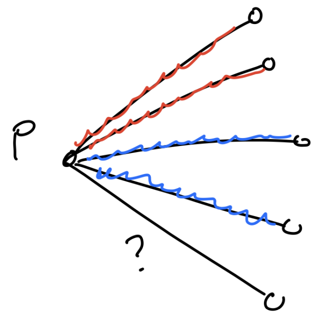
Remark: $K_{3,3}$ is sometimes called the *utility graph*, with three vertices representing houses and the other three representing utilities (water, gas, electric). In the early 20th century, Henry Dudeney posed the *three utilities puzzle*, which asks for a planar drawing connecting each house to each utility. We just showed that this puzzle has no solution. More recently, the puzzle has appeared on coffee mugs. In this case a drawing does exist if you use the handle.

6. Theorem on Friends and Strangers. Consider a complete graph K_6, where each edge is colored either red or blue.

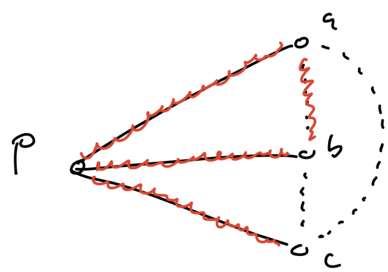
(a) Pick a random vertex p. Show that there exist three other vertices a, b, c so that the edges pa, pb, pc all have the same color. [Hint: There are 5 edges coming out of p.]

(b) Use part (a) to show that the graph contains a red triangle or a blue triangle (or both). [Hint: Suppose that the edges pa, pb, pc are all red. If at least one of the edges ab, ac, bc is red then the graph contains a red triangle. Otherwise . . .]

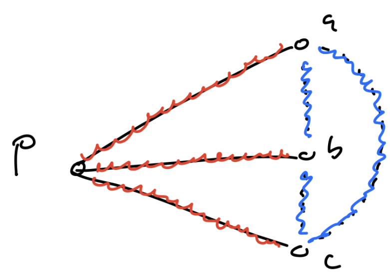
(a): Pick a random vertex p and consider the 5 edges out of p. I claim that there exist three of these edges with the same color. Indeed, if this were not true then we would have ≤ 2 red edges and ≤ 2 blue edges, which don't add up to 5:



(b): Without loss of generality,[3] assume that there are 3 red edges coming out of p. Call them pa, pb and pc. Now consider the three edges ab, ac and bc. If at least one of these edges is red, then we obtain a red triangle:



Otherwise, all three of the edges ab, ac, bc are blue, in which case we must have a blue triangle:



Thus we have shown that a complete graph K_6 with two edge colors must contain a monochromatic triangle.

Remark: A version of *Ramsey's Theorem* says that for any integer $n \geq 1$, there exists a number $R(n)$ such that any bicolored complete graph with at least $R(n)$ edges must have a monochromatic copy of K_n inside it. We just proved that $R(3) = 6$. Wikipedia tells me that $R(4) = 18$. The number $R(5)$ is still unknown! It is somewhere between 43 and 48.

[3]We use this phrase when a proof splits into two cases that are basically equivalent. The other case here is when there are 3 blue edges coming out of p. The proof of this case is exactly the same as the proof given here, except that the colors are switched. It would be tedious to write out the same proof twice.