Math 309 **Fall 2022**
Homework 5 **Drew Armstrong**

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1. Vertex Degrees.

(a) Explain why $1,2,3,4,5,6$ cannot be the vertex degrees of a graph. [Hint: Handshake.]
(b) Explain why $1,1,1,1,1,3$ cannot be the vertex degrees of a connected graph. [Hint: There are $n=6$ vertices. Use Handshaking to find the number of edges e. A connected graph must have $n-1\leq e$.]
(c) Draw a non-connected graph with vertex degrees $1,1,1,1,1,3$.

2. Complete Bipartite Graphs. The complete bipartite graph $K_{m,n}$ consists of $m+n$ vertices $\{u_1,\ldots,u_m,v_1,\ldots,v_n\}$ and mn edges $\{u_i,v_j\}$ for all $1\leq i\leq m$ and $1\leq j\leq n$.

(a) For a given graph G, let G' denote the complementary graph with edges and non-edges switched. Draw pictures of $K_{3,4}$ and its complement $K'_{3,4}$.
(b) For all m and n, explain why

$$(\#\text{ edges in } K_{m,n}) + (\#\text{ edges in } K'_{m,n}) = \binom{m+n}{2}.$$

3. The Hypercube Graph. The hypercube graph Q_n has 2^n vertices, corresponding to the binary strings (words from the alphabet $\{0,1\}$) of length n. We draw an edge between two vertices if the corresponding words differ in a single position.

(a) Draw the graphs Q_1, Q_2 and Q_3.
(b) Compute the number of edges in Q_n. [Hint: Use the Handshaking Lemma. Note that each vertex of Q_n has the same degree.]

4. Tree Degrees. Let T be a tree with n vertices, and let n_k be the number of vertices of degree k, so that $n=\sum_k n_k$.

(a) Explain why $\sum_k k\cdot n_k = 2(n-1)$. [Hint: A tree with n vertices has $n-1$ edges.]
(b) Use part (a) to prove that

$$n_1 = 2 + n_3 + 2n_4 + 3n_5 + 4n_6 + \cdots.$$

(c) A *parrafin molecule* is a saturated, acylic hydrocarbon. We can think of this as a tree with only vertices of degree 1 (hydrogen atoms) and degree 4 (carbon atoms). In this case, use part (b) to show that

$$(\#\text{ of hydrogen atoms}) = 2 + 2(\#\text{ of carbon atoms}).$$

5. Planarity of Bipartite Graphs. Let G be a simple, bipartite graph (i.e., with no loops, no multiple edges, and no cycles of odd length) with v vertices and e edges.

(a) Suppose that G has a planar drawing with f faces. In this case, show that

$$2e\geq 4f.$$

[Hint: By the Handshaking Lemma, the sum of the degrees of the faces equals $2e$. By our assumptions on G, each face in the drawing must have degree ≥ 4.]
(b) Combine (a) with Euler's Formula $v-e+f=2$ to show that

$$e\leq 2v-4.$$

(c) Use part (b) to prove that the complete bipartite graph $K_{3,3}$ has no planar drawing.

6. Theorem on Friends and Strangers. Consider a complete graph K_6, where each edge is colored either red or blue.

(a) Pick a random vertex p. Show that there exist three other vertices a, b, c so that the edges pa, pb, pc all have the same color. [Hint: There are 5 edges coming out of p.]

(b) Use part (a) to show that the graph contains a red triangle or a blue triangle (or both). [Hint: Suppose that the edges pa, pb, pc are all red. If at least one of the edges ab, ac, bc is red then the graph contains a red triangle. Otherwise $\ldots$]