Math 309 **Fall 2022**
Homework 4 **Drew Armstrong**

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1. Write It Down! In each case, explicitly write down all the possibilities.

(a) Ordered selections of 3 things from the set $\{a, b, c, d\}$. No repetition allowed.
(b) Unordered selections of 2 things from the set $\{a, b, c, d, e, f\}$. No repetition allowed.
(c) Non-negative integer solutions $c, v, s \geq 0$ to the equation $c + v + s = 4$. [Hint: There are three flavors of ice cream. You want to buy four gallons.]

(a): There are ${}_4P_3 = 4 \cdot 3 \cdot 2 = 24$ choices:

$$\begin{array}{cccc} abc & abd & acd & bcd \\ acb & adb & adc & bdc \\ bac & bad & cad & cbd \\ bca & bda & cda & cdb \\ cab & dab & dac & dbc \\ cba & dba & dca & dcb \end{array}$$

Remark: There are ${}_4C_3 = \binom{4}{3} = 4$ **unordered** choices:

$$\begin{array}{cccc} abc & abd & acd & bcd \end{array}$$

Note that ${}_4P_3 = {}_4C_3 \cdot 3!$ since there are $3! = 6$ ways to order each unordered choice. More generally, we have ${}_nP_k = {}_nC_k \cdot k!$ for any $0 \leq k \leq n$. This is how we computed ${}_nC_k$:

$${}_nC_k = \frac{1}{k!} \cdot {}_nP_k = \frac{1}{k!}(n)(n-1)\cdots(n-k+1) = \frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

(b): Unordered selections of 2 things from $\{a, b, c, d, e, f\}$ are the same as subsets of size 2. There are $\binom{6}{2} = 15$ such subsets:

$$\begin{array}{lllll} ab & bc & cd & de & ef \\ ac & bd & ce & df & \\ ad & be & cf & & \\ ae & bf & & & \\ af & & & & \end{array}$$

Note: To save space I wrote cd instead of $\{c, d\}$, etc.

(c): A solution to $c + v + s = 4$ with $c, v, s \geq 0$ is the same as a selection of 4 gallons of ice cream from the 3 flavors $\{\text{chocolate}, \text{vanilla}, \text{strawberry}\}$. That is, we are selecting 4 things from 3 things, where repetition is allowed and order doesn't matter. A choice can be encoded as a sequence of "stars and bars", with 4 stars and 2 bars:

$$\underbrace{* \cdots *}_{c \text{ times}} | \underbrace{* \cdots *}_{v \text{ times}} | \underbrace{* \cdots *}_{s \text{ times}}.$$

There are $\binom{6}{2} = \binom{6}{4} = 15$ such sequences:

$$\begin{array}{lllll} ||**** & *||*** & **||** & ***||* & ****|| \\ |*|*** & *|*|** & **|*|* & ***|*| & \\ |**|** & *|**|* & **|**| & & \\ |***|* & *|***| & & & \\ |****| & & & & \end{array}$$

corresponding to 15 solutions for (c, v, s):

$$\begin{array}{lllll} (0,0,4) & (1,0,3) & (2,0,2) & (3,0,1) & (4,0,0) \\ (0,1,3) & (1,1,2) & (2,1,1) & (3,1,0) & \\ (0,2,2) & (1,2,1) & (2,2,0) & & \\ (0,3,1) & (1,3,0) & & & \\ (0,4,0) & & & & \end{array}$$

2. Just the Numbers, Please. Count the possibilities in each case.

(a) Phone numbers consisting of 7 digits.
(b) Rearrangements of the letters m, a, m, m, a, l.
(c) Poker hands, consisting of 5 cards drawn from a deck of 52.
(d) Non-negative integer solutions $x + y + z \geq 0$ to the equation $x + y + z = 7$.

(a): The number of 7-digit phone numbers is

$$\underbrace{10}_{\text{1st digit}} \times \underbrace{10}_{\text{2nd digit}} \times \cdots \times \underbrace{10}_{\text{7th digit}} = 10^7.$$

(b): The number of arrangements of the letters m, a, m, m, a, l is

$$\frac{6!}{3!2!1!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 1} = 60.$$

Remark: More generally, the number of words of length n containing k_1 copies of the letter a_1, k_2 copies of the letter a_2, ..., and k_ℓ copies of the letter a_ℓ is the multinomial coefficient:

$$\binom{n}{k_1, k_2, \ldots, k_\ell} = \frac{n!}{k_1! k_2! \cdots k_\ell!}.$$

When using this notation we always assume that $k_1 + k_2 + \cdots + k_\ell = n$.

(c): A poker hand is a collection of 5 unordered cards, chosen without replacement from a deck of 52. The number of choices is

$$\binom{52}{5} = \frac{52!}{5!47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2,598,960.$$

(d): Compare to Problem 1(c). A non-negative integer solution to the equation $x + y + z = 7$ corresponds to a sequence of 7 stars and 2 bars. The number of such sequences is

$$\binom{9}{2} = \binom{9}{7} = \frac{9!}{2!7!} = \frac{9 \cdot 8}{2 \cdot 1} = 36.$$

Remark: More generally, a non-negative integer solution to $x_1 + \cdots + x_n = k$ corresponds to a sequence of k stars and $n - 1$ bars. The number of such sequences is

$$\binom{k + (n-1)}{k, n-1} = \binom{n + k - 1}{k} = \binom{n + k - 1}{n - 1} = \cdots$$

The previous calculation corresponds to $k = 7$ and $n = 3$.

3. Vandermonde Convolution. For any positive integers r, g, n we have[1]

$$\sum_k \binom{r}{k} \binom{g}{n-k} = \binom{r+g}{n}.$$

[1] We sum over all integers k, but only finitely many summands will be non-zero.

(a) Give a counting proof of this identity. [Hint: There are r red balls and g green balls in a bowl. You reach in and grab a collection of n unordered balls.]
(b) Use the identity to prove that $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$.

(a): There are r red balls and g green balls in a bowl. Let S be the **set of possible choices** of n balls from the bowl. On the one hand we have

$$\#S = \binom{r+g}{n}.$$

On the other hand, let S_k be the **set of possible choices** consisting of k red balls and $n-k$ green balls. Note that we have a disjoint union:

$$S = S_0 \cup S_1 \cup \cdots \cup S_r.$$

Indeed, each choice of n balls contains **some number of red balls**. It follows that

$$\#S = \#S_0 + \#S_1 + \cdots + \#S_r.$$

But we also have

$$\#S_k = \underbrace{\binom{r}{k}}_{\substack{\text{number of ways to}\\ \text{choose } k \text{ red balls}}} \times \underbrace{\binom{g}{n-k}}_{\substack{\text{number of ways to}\\ \text{choose } n-k \text{ green balls}}} .$$

We conclude that

$$\binom{r+g}{n} = \sum_k \binom{r}{k}\binom{g}{n-k}.$$

Only finitely many terms in the sum are nonzero.[2]

4. Trinomial Recurrence. The trinomial coefficients are defined as follows:

$$\binom{n}{i,j,k} := \frac{n!}{i!j!k!}, \qquad \text{where we must have } i+j+k=n.$$

Use algebra to prove the *trinomial recurrence relation*:

$$\binom{n}{i,j,k} = \binom{n-1}{i-1,k,j} + \binom{n-1}{i,j-1,k} + \binom{n-1}{i,j,k-1}.$$

We will repeatedly use the fact that $m(m-1)! = m!$. Note that

$$\begin{aligned}
&\binom{n-1}{i-1,k,j} + \binom{n-1}{i,j-1,k} + \binom{n-1}{i,j,k-1}\\
&= \frac{(n-1)!}{(i-1)!j!k!} + \frac{(n-1)!}{i!(j-1)!k!} + \frac{(n-1)!}{i!j!(k-1)!}\\
&= \frac{i}{i}\cdot\frac{(n-1)!}{(i-1)!j!k!} + \frac{j}{j}\cdot\frac{(n-1)!}{i!(j-1)!k!} + \frac{k}{k}\cdot\frac{(n-1)!}{i!j!(k-1)!}\\
&= \frac{i(n-1)!}{i(i-1)!j!k!} + \frac{j(n-1)!}{i!j(j-1)!k!} + \frac{k(n-1)!}{i!j!k(k-1)!}\\
&= \frac{i(n-1)!}{i!j!k!} + \frac{j(n-1)!}{i!j!k!} + \frac{k(n-1)!}{i!j!k!}
\end{aligned}$$

[2] Recall that we define $\binom{a}{b} = 0$ when $b < 0$ or $b > a$. Without this notational convenience, we must specify that $0 \le k$, $k \le r$, $k \le n$ and $n-k \le g$, so that $\max\{0, n-g\} \le k \le \min\{r, n\}$, which is quite annoying to say.

$$\begin{aligned}
&= \frac{i(n-1)! + j(n-1)! + k(n-1)!}{i!j!k!} \\
&= \frac{(i+j+k)(n-1)!}{i!j!k!} \\
&= \frac{n(n-1)!}{i!j!k!} \\
&= \frac{n!}{i!j!k!} \\
&= \binom{n}{i,j,k}.
\end{aligned}$$

5. Double Factorial. For a positive integer n we define the *double factorial* as follows:

$$n!! = \begin{cases} n(n-2)(n-4)\cdots 4\cdot 2 & \text{if } n \text{ is even,} \\ n(n-2)(n-4)\cdots 3\cdot 1 & \text{if } n \text{ is odd.} \end{cases}$$

(a) For any $m \geq 1$, show that $(2m)!!(2m-1)!! = (2m)!$.
(b) For any $m \geq 1$, show that $(2m)!! = 2^m m!$.
(c) Combine (a) and (b) to show that $(2m-1)!! = \frac{(2m)!}{2^m m!}$.

(a): We have

$$\begin{aligned}
(2m)! &= (2m)(2m-1)(2m-2)(2m-3)\cdots 4\cdot 3\cdot 2\cdot 1 \\
&= [(2m)(2m-2)\cdots 4\cdot 2][(2m-1)(2m-3)\cdot\cdots 3\cdot 1] \\
&= (2m)!!(2m-1)!!.
\end{aligned}$$

(b): We have

$$\begin{aligned}
(2m)!! &= (2m)(2m-2)(2m-4)\cdots 4\cdot 2 \\
&= [2(m)][2(m-1)][2(m-2)]\cdots[2(2)][2(1)] \\
&= 2^m(m)(m-1)(m-2)\cdot\cdots 2\cdot 1 \\
&= 2^m m!.
\end{aligned}$$

(c): We have

$$\begin{aligned}
(2m)!!(2m-1)!! &= (2m)! && \text{(a)} \\
(2m-1)!! &= (2m)!/(2m)!! \\
(2m-!)!! &= \frac{(2m)!}{2^m m!}. && \text{(b)}
\end{aligned}$$

6. Generalized Binomial Coefficients. For any number z and positive integer k we define

$$\binom{z}{k} = \frac{(z)_k}{k!} = \frac{z(z-1)\cdots(z-k+1)}{k!}.$$

This formula agrees with the usual binomial coefficients when z is a positive integer, but it makes sense even when z is negative or when z is a fraction.

(a) Use the formula to compute $\binom{-3}{4}$.

(b) Give an algebraic proof that

$$\binom{-z}{k} = (-1)^k \binom{z+k-1}{k}.$$

(c) Give an algebraic proof that

$$\binom{1/2}{k} = \frac{(-1)^{k-1}}{k \cdot 2^{2k-1}} \cdot \binom{2(k-1)}{k-1}.$$

[Hint: At some point you will need to use Problem 5(c) with $m = k - 1$.]

(a): We have

$$\binom{-3}{4} = \frac{(-3)_4}{4!} = \frac{(-3)(-4)(-5)(-6)}{4 \cdot 3 \cdot 2 \cdot 1} = 15.$$

(b): We have

$$\begin{aligned}
\binom{-n}{k} &= \frac{(-n)_k}{k!} \\
&= \frac{1}{k!}(-n)(-n-1)(-n-2)\cdots(-n-k+1) \\
&= \frac{1}{k!}[(-1)(n)][(-1)(n+1)][(-1)(n+2)]\cdots[(-1)(n+k-1)] \\
&= \frac{(-1)^k}{k!}(n)(n+1)(n+2)\cdots(n+k-1) \\
&= \frac{(-1)^k}{k!}(n+k-1)(n+k-2)\cdots(n+1)(n) \\
&= \frac{(-1)^k}{k!}(n+k-1)_k \\
&= (-1)^k \frac{(n+k-1)_k}{k!} \\
&= (-1)^k \binom{n+k-1}{k}.
\end{aligned}$$

(c): We have

$$\begin{aligned}
\binom{1/2}{k} &= \frac{1}{k!}(1/2)_k \\
&= \frac{1}{k!}(1/2)(1/2-1)(1/2-2)\cdots(1/2-k+1) \\
&= \frac{1}{k!}(1/2)(-1/2)(-3/2)\cdots((-2k+3)/2) \\
&= \frac{1}{k!}[(1/2)(1)][(1/2)(-1)][(1/2)(-3)]\cdots[(1/2)(-2k+3)] \\
&= \frac{1}{k!}\left(\frac{1}{2}\right)^k (1)(-1)(-3)\cdots(-2k+3) \\
&= \frac{1}{k!}\left(\frac{1}{2}\right)^k (-1)(-3)\cdots(-(2(k-1)-1))
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{k!}\left(\frac{1}{2}\right)^k (-1)^{k-1}(1)(3)\cdots(2(k-1)-1) \\
&= \frac{(-1)^{k-1}}{2^k k!}(1)(3)\cdots(2(k-1)-1) \\
&= \frac{(-1)^{k-1}}{2^k k!}(2(k-1))-1)\cdots(3)(1) \\
&= \frac{(-1)^{k-1}}{2^k k!}(2(k-1)-1)!! \\
&= \frac{(-1)^{k-1}}{2^k k!}\cdot\frac{(2(k-1))!}{2^{k-1}(k-1)!} && \text{Problem 5(c)} \\
&= \frac{(-1)^{k-1}}{2^k 2^{k-1} k(k-1)!}\cdot\frac{(2(k-1))!}{(k-1)!} \\
&= \frac{(-1)^{k-1}}{k\cdot 2^{2k-1}}\cdot\frac{(2(k-1))!}{(k-1)!(k-1)!} \\
&= \frac{(-1)^{k-1}}{k\cdot 2^{2k-1}}\cdot\binom{2(k-1)}{k-1}.
\end{aligned}
$$

That was fun.

Remark: Combining this calculation with Newton's binomial theorem gives us the power series expansion of $\sqrt{1+x}$ for $|x|<1$:

$$
\begin{aligned}
\sqrt{1+x} &= (1+x)^{1/2} \\
&= \sum_{k\geq 0}\binom{1/2}{k}\cdot x^k \\
&= \sum_{k\geq 0}\frac{(-1)^{k-1}}{k\cdot 2^{2k-1}}\cdot\binom{2(k-1)}{k-1}\cdot x^k \\
&= 1+\frac{1}{2}\cdot x-\frac{1}{8}\cdot x^2+\frac{1}{16}\cdot x^3-\frac{5}{128}\cdot x^4+\frac{7}{256}\cdot x^5-\frac{21}{1024}\cdot x^6+\cdots.
\end{aligned}
$$