Math 309 **Fall 2022**
Homework 4 **Drew Armstrong**

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1. Write It Down! In each case, explicitly write down all the possibilities.

(a) Ordered selections of 3 things from the set $\{a, b, c, d\}$. No repetition allowed.
(b) Unordered selections of 2 things from the set $\{a, b, c, d, e, f\}$. No repetition allowed.
(c) Non-negative integer solutions $c, v, s \geq 0$ to the equation $c + v + s = 4$. [Hint: There are three flavors of ice cream. You want to buy four gallons.]

2. Just the Numbers, Please. Count the possibilities in each case.

(a) Phone numbers consisting of 7 digits.
(b) Rearrangements of the letters m, a, m, m, a, l.
(c) Poker hands, consisting of 5 cards drawn from a deck of 52.
(d) Non-negative integer solutions $x + y + z \geq 0$ to the equation $x + y + z = 7$.

3. Vandermonde Convolution. For any positive integers r, g, n we have[1]

$$\sum_k \binom{r}{k}\binom{g}{n-k} = \binom{r+g}{n}.$$

(a) Give a counting proof of this identity. [Hint: There are r red balls and g green balls in a bowl. You reach in and grab a collection of n unordered balls.]
(b) Use the identity to prove that $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$.

4. Trinomial Recurrence. The trinomial coefficients are defined as follows:

$$\binom{n}{i, j, k} := \frac{n!}{i!j!k!}, \qquad \text{where we must have } i + j + k = n.$$

Use algebra to prove the *trinomial recurrence relation*:

$$\binom{n}{i, j, k} = \binom{n-1}{i-1, k, j} + \binom{n-1}{i, j-1, k} + \binom{n-1}{i, j, k-1}.$$

5. Double Factorial. For a positive integer n we define the *double factorial* as follows:

$$n!! = \begin{cases} n(n-2)(n-4)\cdots 4 \cdot 2 & \text{if } n \text{ is even,} \\ n(n-2)(n-4)\cdots 3 \cdot 1 & \text{if } n \text{ is odd.} \end{cases}$$

(a) For any $m \geq 1$, show that $(2m)!!(2m-1)!! = (2m)!$.
(b) For any $m \geq 1$, show that $(2m)!! = 2^m m!$.
(c) Combine (a) and (b) to show that $(2m-1)!! = \frac{(2m)!}{2^m m!}$.

6. Generalized Binomial Coefficients. For any number z and positive integer k we define

$$\binom{z}{k} = \frac{(z)_k}{k!} = \frac{z(z-1)\cdots(z-k+1)}{k!}.$$

This formula agrees with the usual binomial coefficients when z is a positive integer, but it makes sense even when z is negative or when z is a fraction.

(a) Use the formula to compute $\binom{-3}{4}$.

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[1] We sum over all integers k, but only finitely many summands will be non-zero.

(b) Give an algebraic proof that

$$\binom{-z}{k} = (-1)^k \binom{z+k-1}{k}.$$

(c) Give an algebraic proof that

$$\binom{1/2}{k} = \frac{(-1)^{k-1}}{k \cdot 2^{2k-1}} \cdot \binom{2(k-1)}{k-1}.$$

[Hint: At some point you will need to use Problem 5(c) with $m = k-1$.]