Math 309 Homework 2 — Fall 2022 Drew Armstrong

1. Truth Tables.

(a) Draw truth tables to prove that $\neg(P \vee Q) = \neg P \wedge \neg Q$ and $\neg(P \wedge Q) = \neg P \vee \neg Q$.
(b) Draw truth tables for the following four Boolean functions:

$$P \Rightarrow Q \qquad Q \Rightarrow P \qquad \neg P \Rightarrow \neg Q \qquad \neg Q \Rightarrow \neg P.$$

Which ones are the same?

(a): Note that columns 4 and 7 are the same.

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

The equivalence of $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$ is called *de Morgan's Law.*

(b): Note that columns 5,8 are the same, and columns 6,7 are the same, but columns 5,8 are not the same as 6,7:

P	Q	$\neg P$	$\neg Q$	$P \Rightarrow Q$	$Q \Rightarrow P$	$\neg P \Rightarrow \neg Q$	$\neg Q \Rightarrow \neg P$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

The equivalence of $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$ is called the *principle of the contrapositive.*

2. Methods of Proof.

(a) Prove that $P \Rightarrow (R \vee Q)$ equals $(\neg Q \wedge \neg R) \Rightarrow \neg P$. [Hint: You could use a truth table but it's easier to combine Problem 1(ab) using algebra.]
(b) Use part (a) to prove the following theorem about integers $n, m \in \mathbb{Z}$:

If mn is even, then m is even or n is even.

[Hint: Name the statements P, Q, R.]

(a): Combining Problems 1(ab) gives

$$\begin{aligned} P \Rightarrow (Q \vee R) &= \neg(Q \vee R) \Rightarrow \neg P && \text{1(b)} \\ &= (\neg Q \wedge \neg R) \Rightarrow \neg P. && \text{1(a)} \end{aligned}$$

(b): Let $m, n \in \mathbb{Z}$ and consider the following statements:

$$\begin{aligned} P &= \text{“}mn \text{ is even”}, \\ Q &= \text{“}m \text{ is even”}, \\ R &= \text{“}n \text{ is even”}. \end{aligned}$$

Our goal is to prove that $P \Rightarrow (Q \vee R)$. It is difficult to prove this directly, so we will instead prove the equivalent statement $(\neg Q \wedge \neg R) \Rightarrow \neg P$:

If m and n are both odd, then mn is odd.

To prove this, consider any odd integers $m, n \in \mathbb{Z}$. By definition this means that $m = 2k+1$ and $n = 2\ell + 1$ for some integers $k, \ell \in \mathbb{Z}$. But then we have

$$\begin{aligned} mn &= (2k+1)(2\ell+1) \\ &= 4k\ell + 2k + 2\ell + 1 \\ &= 2(2k\ell + k + \ell) + 1 \\ &= 2(\text{some integer}) + 1, \end{aligned}$$

which shows that mn is odd. $\square$

3. Peirce's Arrow. The operator NOR (also called *Peirce's arrow*) is defined as follows:

$$P \text{ NOR } Q = P \downarrow Q = \text{NOT } (P \text{ OR } Q) = \neg(P \vee Q).$$

Use Boolean algebra (i.e., don't use truth tables) to prove the following identities.

(a) $\neg P = P \downarrow P$
(b) $P \wedge Q = (P \downarrow P) \downarrow (Q \downarrow Q)$
(c) $P \vee Q = (P \downarrow Q) \downarrow (P \downarrow Q)$

(a): We have

$$\begin{aligned} P \downarrow P &= \neg(P \vee P) && \text{definition} \\ &= \neg P. && A \vee A = A \end{aligned}$$

(b): We have

$$\begin{aligned} (P \downarrow P) \downarrow (Q \downarrow Q) &= (\neg P) \downarrow (\neg Q) && \text{part (a)} \\ &= \neg(\neg P \vee \neg Q) && \text{definition} \\ &= \neg(\neg P) \wedge \neg(\neg Q) && \text{de Morgan} \\ &= P \wedge Q. && \neg\neg A = A \end{aligned}$$

(a): We have

$$\begin{aligned} (P \downarrow Q) \downarrow (P \downarrow Q) &= \neg(P \vee Q) \downarrow \neg(P \vee Q) && \text{definition} \\ &= \neg(\neg(P \vee Q) \vee \neg(P \vee Q)) && \text{definition} \\ &= \neg\neg(P \vee Q) \wedge \neg\neg(P \vee Q) && \text{de Morgan} \\ &= (P \vee Q) \wedge (P \vee Q) && \neg\neg A = A \\ &= P \vee Q. && A \wedge A = A. \end{aligned}$$

Remark: There are infinitely correct proofs. For example, you could make the expression look more and more complicated, before finally simplifying it. In general we like to have short proofs, but sometimes there are two different proofs of the same length and it's hard to choose between them. In class I gave a different proof of part (c).

4. Injective, Surjective, Bijective. Let $f : S \to T$ be a function of finite sets. For each element $t \in T$ we consider the number

$$d(t) := \#\{s \in S : f(s) = t\} = \text{ the number of elements of } S \text{ that get sent to } t.$$

We say that f is *injective* if $d(t) \leq 1$ for all $t \in T$, *surjective* if $d(t) \geq 1$ for all $t \in T$ and *bijective* if $d(t) = 1$ for all $t \in T$.

(a) If f is injective, prove that $\#S \leq \#T$.
(b) If f is surjective, prove that $\#S \geq \#T$.
(c) If f is bijective, prove that $\#S = \#T$.

[Hint: Observe that $\#S = \sum_{t \in T} d(t)$ and $\#T = \sum_{t \in T} 1$.]

The key to the whole problem is to observe that $\sum_{t \in T} d(t) = \#S$. Indeed, since $d(t)$ is the number of arrows pointing to t, the sum on the left counts all of the arrows in the function. On the other hand, since each element of S has exactly one arrow (by the definition of function), the number of arrows is also equal to $\#S$.

(a): Suppose that $f : S \to T$ is injective, so that $d(t) \leq 1$ for all $t \in T$. It follows that

$$\#S = \sum_{t \in T} d(t) \leq \sum_{t \in T} 1 = \#T.$$

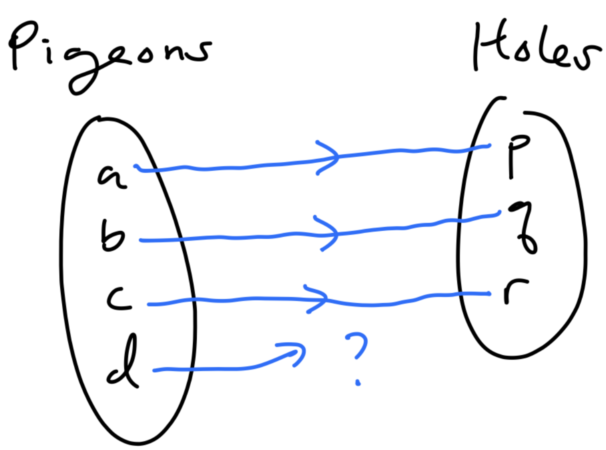
(b): Suppose that $f : S \to T$ is surjective, so that $d(t) \geq 1$ for all $t \in T$. It follows that

$$\#S = \sum_{t \in T} d(t) \geq \sum_{t \in T} 1 = \#T.$$

(c): Suppose that $f : S \to T$ is bijective, so that $d(t) = 1$ for all $t \in T$. It follows that

$$\#S = \sum_{t \in T} d(t) = \sum_{t \in T} 1 = \#T.$$

Remark: This theorem is sometimes associated with the *pigeonhole principle*. For example, any function from a set of 4 elements to a set of 3 elements must be non-injective. In other words, since there are 4 pigeons and 3 pigeonholes, two pigeons must share a hole:



5. Counting Functions. Compute the number of each kind of function.

(a) All functions from $\{1,2,3,4,5\} \to \{1,2\}$.
(b) Injective functions from $\{1,2\}$ to $\{1,2,3,4,5\}$.
(c) Surjective functions from $\{1,2\}$ to $\{1,2,3,4,5\}$.
(d) Surjective functions from $\{1,2,3,4,5\}$ to $\{1,2\}$. [Hint: In how many ways can you choose the subset of $\{1,2,3,4,5\}$ that get sent to 1? You can't send everything to 1 and you can't send nothing to 1.]

Recall that the number of functions from S to T is $\#T^{\#S}$. Indeed, to define a function $f : S \to T$ we must choose an element $f(s) \in T$ for each element $s \in S$. There are $\#T$ possible choices of $f(s)$ for each element $s \in S$ so the total number of functions is

$$\#(\text{functions } S \to T) = \underbrace{\#T \times \#T \times \cdots \times \#T}_{\#S \text{ times}} = \#T^{\#S}.$$

(a): From the previous remark, the number of functions from $\{1,2,3,4,5\}$ to $\{1,2\}$ is $2^5 = 32$. Remark: Such functions are equivalent to subsets of $\{1,2,3,4,5\}$. For example, the subset $\{1,3,4\} \subseteq \{1,2,3,4,5\}$ corresponds to the function that sends $1,3,4$ to 1 and $2,5$ to 2.[1]

(b): The total number of functions from $\{1,2\}$ to $\{1,2,3,4,5\}$ is $5^2 = 25$. How many of these are injective? To choose an injective function $f : \{1,2\} \to \{1,2,3,4,5\}$ we will first choose the value $f(1)$. There are 5 ways to do this. Then, having chosen $f(1)$ we need to choose the value of $f(2)$. If f is injective then we are not allowed to have $f(1) = f(2)$ so we must choose $f(2)$ from the remaining 4 elements. The total number of choices is

$$\#(\text{injective functions } \{1,2\} \to \{1,2,3,4,5\}) = \underbrace{5}_{\text{choose } f(1)} \times \underbrace{4}_{\text{then choose } f(2)} = 20.$$

Remark: Using the same reasoning, we have

$$\#(\text{injective functions } \{1,2,\ldots,k\} \to \{1,2,\ldots,n\}) = n(n-1)(n-2)\cdots(n-k+1).$$

If $k > n$ then this number is zero, in accordance with Problem 4(a).

(c): There are zero surjective functions from $\{1,2\}$ to $\{1,2,3,4,5\}$. Indeed, if there exists a surjective function $S \to T$ then Problem 4(b) says that $\#S \geq \#T$. Equivalently, if $\#S < \#T$ then do not exist any surjective functions $S \to T$. [This is another example of the contrapositive.]

(d): We saw in part (a) that the total number of functions from $\{1,2,3,4,5\}$ to $\{1,2\}$ is $2^5 = 32$. How many of these are surjective? It is easier to count non-surjective functions. If $f : \{1,2,3,4,5\} \to \{1,2\}$ is not surjective then one of the elements $\{1,2\}$ does not get hit by f. If 2 does not get hit then f must send every element to 1. If 1 does not get hit then f must send every element to 2. Thus there are exactly two non-surjective functions. It follows that

$$\begin{aligned}\#(\text{surjective functions } \{1,2,3,4,5\} \to \{1,2\}) &= \#(\text{functions}) - \#(\text{non-surjective functions}) \\ &= 32 - 2 \\ &= 30.\end{aligned}$$

Remarks:

[1] Or you could send $1,3,4$ to 2 and $2,5$ to 1. It doesn't matter which; you just need to pick an element $x \in \{1,2\}$ and say that elements of $\{1,2,3,4,5\}$ sent to x are in the corresponding subset.

- It follows from Problem 4(a) that there are no injective functions $\{1,2,3,4,5\} \to \{1,2\}$. Indeed, if there exists an injective function $S \to T$ then 4(a) says that $\#S \leq \#T$. Equivalently, if $\#S > \#T$ then there do not exist any injective functions $S \to T$.
- It is quite difficult to count surjective functions $\{1,2,\ldots,k\} \to \{1,2,\ldots,n\}$. The *Stiling numbers of the second kind* are defined by the following recurrence:[2]

$$S(k,n) = \begin{cases} 1 & k = 0 \text{ and } n = 0, \\ 0 & k = 0 \text{ or } n = 0 \text{ but not both}, \\ nS(k-1,n) + S(k-1,n-1) & k \geq 1 \text{ and } n \geq 1. \end{cases}$$

  One can show that the number of surjective functions $\{1,2,\ldots,k\} \to \{1,2,\ldots,n\}$ is $n! \cdot S(k,n)$. For example, when $k = 5$ and $n = 2$ we showed that there are 30 surjective functions, which means that $30 = 2! \cdot S(5,2)$ and hence $S(5,2) = 15$.

[2]Horrible notation. There are also Stirling numbers of the first kind but I don't want to talk about them.