Math 309 **Fall 2022**
Homework 2 **Drew Armstrong**

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1. Truth Tables.

(a) Draw truth tables to prove that $\neg(P \vee Q) = \neg P \wedge \neg Q$ and $\neg(P \wedge Q) = \neg P \vee \neg Q$.
(b) Draw truth tables for the following four Boolean functions:

$$P \Rightarrow Q \qquad Q \Rightarrow P \qquad \neg P \Rightarrow \neg Q \qquad \neg Q \Rightarrow \neg P.$$

Which ones are the same?

2. Methods of Proof.

(a) Prove that $P \Rightarrow (R \vee Q)$ equals $(\neg Q \wedge \neg R) \Rightarrow \neg P$. [Hint: You could use a truth table but it's easier to combine Problem 1(ab) using algebra.]
(b) Use part (a) to prove the following theorem about integers $n, m \in \mathbb{Z}$:

If mn is even, then m is even or n is even.

[Hint: Name the statements P, Q, R.]

3. Peirce's Arrow. The operator NOR (also called *Peirce's arrow*) is defined as follows:

$$P \text{ NOR } Q = P \downarrow Q = \text{NOT } (P \text{ OR } Q) = \neg(P \vee Q).$$

Use Boolean algebra (i.e., don't use truth tables) to prove the following identities.

(a) $\neg P = P \downarrow P$
(b) $P \wedge Q = (P \downarrow P) \downarrow (Q \downarrow Q)$
(c) $P \vee Q = (P \downarrow Q) \downarrow (P \downarrow Q)$

4. Injective, Surjective, Bijective. Let $f : S \to T$ be a function of finite sets. For each element $t \in T$ we consider the number

$$d(t) := \#\{s \in S : f(s) = t\} = \text{ the number of elements of } S \text{ that get sent to } t.$$

We say that f is *injective* if $d(t) \leq 1$ for all $t \in T$, *surjective* if $d(t) \geq 1$ for all $t \in T$ and *bijective* if $d(t) = 1$ for all $t \in T$.

(a) If f is injective, prove that $\#S \leq \#T$.
(b) If f is surjective, prove that $\#S \geq \#T$.
(c) If f is bijective, prove that $\#S = \#T$.

[Hint: Observe that $\#S = \sum_{t \in T} d(t)$ and $\#T = \sum_{t \in T} 1$.]

5. Counting Functions. Compute the number of each kind of function.

(a) All functions from $\{1, 2, 3, 4, 5\} \to \{1, 2\}$.
(b) Injective functions from $\{1, 2\}$ to $\{1, 2, 3, 4, 5\}$.
(c) Surjective functions from $\{1, 2\}$ to $\{1, 2, 3, 4, 5\}$.
(d) Surjective functions from $\{1, 2, 3, 4, 5\}$ to $\{1, 2\}$. [Hint: In how many ways can you choose the subset of $\{1, 2, 3, 4, 5\}$ that get sent to 1? You can't send everything to 1 and you can't send nothing to 1.]