Math 309 **Fall 2022**
Homework 1 **Drew Armstrong**

We will use the following notations for sums of (all, even, odd) pth powers:

$$S_p(n) = 1^p + 2^p + \cdots + n^p = \sum_{k=1}^{n} k^p,$$

$$SE_p(n) = 2^p + 4^p + 6^p + \cdots + (2n)^p = \sum_{k=1}^{n} (2k)^p,$$

$$SO_p(n) = 1^p + 3^p + 5^p + \cdots + (2n-1)^p = \sum_{k=1}^{n} (2k-1)^p.$$

1. Consider the following statement $P(n) =$ "$S_3(n) = n^2(n+1)^2/4$". In this problem you will prove by induction that $P(n)$ is true for all integers $n \geq 1$.

(a) Check by hand that $P(n)$ is true for $n = 1, 2, 3, 4$.

(b) Now fix some arbitrary $n \geq 1$ and **assume for induction** that $P(n)$ is a true statement. In this case, prove that $P(n+1)$ is also a true statement. [Hint: Use the recurrence $S_3(n+1) = S_3(n) + (n+1)^3$.]

(a): For induction we only need to check one base case, but for fun we'll check four:

$$P(1) = \text{“}1^3 = \frac{1^2 \cdot 2^2}{4}\text{”} = \text{“}1 = 1\text{”} = T,$$

$$P(2) = \text{“}1^3 + 2^3 = \frac{2^2 \cdot 3^2}{4}\text{”} = \text{“}9 = 9\text{”} = T,$$

$$P(3) = \text{“}1^3 + 2^3 + 3^3 = \frac{3^2 \cdot 4^2}{4}\text{”} = \text{“}36 = 36\text{”} = T,$$

$$P(4) = \text{“}1^3 + 2^3 + 3^3 + 4^3 = \frac{4^2 \cdot 5^2}{4}\text{”} = \text{“}100 = 100\text{”} = T.$$

(b): Now **fix** some arbitrary integer $n \geq 1$ and **assume** for induction that $P(n)$ is a true statement. That is, suppose that

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

In this case, we must have

$$\begin{aligned}
1^3 + 2^3 + \cdots + (n+1)^3 &= \left[1^3 + 2^3 + \cdots + n^3\right] + (n+1)^3 \\
&= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
&= (n+1)^2 \left[\frac{n^2}{4} + (n+1)\right] \\
&= \frac{(n+1)^2(n^2+4n+4)}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4}.
\end{aligned}$$

Hence the statement $P(n+1)$ is also true. □

2. Find explicit formulas for $SE_2(n)$ and $SO_2(n)$. [Hint: You may assume that $\sum_{k=1}^n k = n(n+1)/2$ and $\sum_{k=1}^n k^2 = n(n+1)(2n+1)/6$.]

The sum of square of even numbers is

$$SE_2(n) = \sum_{k=1}^n (2k)^2 = \sum_{k=1}^n 4k^2 = 4\sum_{k=1}^n k^2 = 4 \cdot \frac{n(n+1)(2n+1)}{6}.$$

The sum of square of odd numbers is

$$\begin{aligned} SO_2(n) &= \sum_{k=1}^n (2k-1)^2 \\ &= \sum_{k=1}^n (4k^2 - 4k + 1) \\ &= 4\sum_{k=1}^n k^2 - 4\sum_{k=1}^n k + \sum_{k=1}^n 1 \\ &= 4 \cdot \frac{n(n+1)(2n+1)}{6} - 4 \cdot \frac{n(n+1)}{2} + n. \end{aligned}$$

You don't need to simplify this, but it turns out that

$$SO_2(n) = \frac{n(2n+1)(2n-1)}{3}.$$

Remark: Observe that

$$\begin{aligned} SE_2(n) + SO_2(n) &= \frac{4n(n+1)(2n+1)}{6} + \frac{2n(2n+1)(2n-1)}{6} \\ &= \frac{(2n)(2n+1)}{6} \cdot [2(n+1) + (2n-1)] \\ &= \frac{(2n)(2n+1)}{6} \cdot [2(2n) + 1] \\ &= S_2(2n), \end{aligned}$$

as it should be.

3. Define the sequence $C_0, C_1, C_2, C_3 \dots$ by the following initial condition and recurrence:

$$C_n := \begin{cases} 1 & \text{if } n = 0, \\ C_{n-1} + n^2 - n & \text{if } n \geq 1. \end{cases}$$

Find a closed formula for C_n.

Write out the first few terms until you see a pattern:

$$\begin{aligned} C_1 &= C_0 + 1^2 - 1, \\ C_2 &= C_1 + 2^2 - 2 = C_0 + 1^2 - 1 + 2^2 - 2, \\ C_3 &= C_2 + 3^3 - 3 = C_0 + 1^2 - 1 + 2^2 - 2 + 3^3 - 3. \end{aligned}$$

We observe that the pattern is

$$C_n = C_0 + 1^1 - 1 + 2^2 - 2 + 3^3 - 3 + \cdots + n^2 - n = C_0 + \sum_{k=1}^n k^2 - \sum_{k=1}^n k.$$

Using the known formulas for sums of squares and first powers, this becomes[1]

$$C_n = C_0 + \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = C_0 + \frac{1}{3}n^3 - \frac{1}{3}n.$$

From this we see that the value of C_0 isn't really important to the general formula.

Remark: Observe that this formula has a nice factorization:

$$C_n = C_0 + \frac{n(n-1)(n+1)}{3}.$$

This was an accident on my part. It follows from specific case of the "hockey stick identity":

$$\sum_{k=1}^{n} \binom{k}{2} = \binom{n+1}{3}.$$

Then we have

$$\sum_{k=1}^{n}(k^2 - k) = 2\sum_{k=1}^{n} \frac{k(k-1)}{2} = 2\sum_{k=1}^{n} \binom{k}{2} = 2\binom{n+1}{3} = \frac{(n+1)n(n-1)}{3}.$$

You don't need to know this.

4. The sequence of *factorials* $0!, 1!, 2!, \ldots$ is defined as follows:

$$n! := \begin{cases} 1 & \text{if } n = 0, \\ (n-1)! \cdot n & \text{if } n \geq 1. \end{cases}$$

You will prove by induction that $n! > 3^n$ for all $n \geq 7$.

(a) Verify that $7! > 3^7$.

(b) Now fix some arbitrary $n \geq 7$ and assume for induction that $n! > 3^n$. In this case, prove that $(n+1)! > 3^{n+1}$. [Hint: Use the facts $(n+1)! = n! \cdot (n+1)$ and $n+1 > 3$.]

(a): My computer says that $7! = 5040$ and $3^7 = 2187$, hence $7! > 3^7$.

(b): Now **fix** some arbitrary $n \geq 7$ and **assume** for induction that $n! > 3^n$. In this case we will show that $(n+1)! > 3^{n+1}$. Indeed, we observe that

$$\begin{aligned} (n+1)! &= (n+1)n! && \text{definition of factorial} \\ &> (n+1)3^n && \text{because } n! > 3^n \\ &> 3 \cdot 3^n && \text{because } n+1 > 3 \\ &= 3^{n+1}. \end{aligned}$$

□

Recall the definition of Pascal's Triangle. For all integers n, k with $n \geq 0$ we have

$$\binom{n}{k} := \begin{cases} 1 & n = 0, k = 0, \\ 0 & n = 0, k \neq 0, \\ \binom{n-1}{k-1} + \binom{n-1}{k} & n \geq 1, k = \text{anything}. \end{cases}$$

[1] You don't need to simplify it.

This definition implies that $\binom{n}{k} = 0$ for $k < 0$ or $k > n$ and $\binom{n}{k} = 1$ for $k = 0$ or $k = n$. The Binomial Theorem says that for all numbers x we have

$$(1+x)^n = \sum_k \binom{n}{k} x^k.$$

5. Use the Binomial Theorem to prove the following identity for all $n \geq 1$:

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n \cdot 2^{n-1}.$$

[Hint: Differentiate with respect to x.]

The Binomial Theorem holds for any value of x:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.$$

Taking the derivative of both sides with respect to x gives

$$n(1+x)^{n=1} = \binom{n}{1}x + \binom{n}{2}(2x) + \cdots + \binom{n}{n}(nx^{n-1}).$$

This formula also holds for any value of x. In particular, substituting $x = 1$ gives

$$n(1+1)^{n=1} = \binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n},$$

which is the formula we want.

6. Let $R_n(d)$ be the maximum number of d-dimensional regions formed by n hyperplanes in d-dimensional space.[2] Ludwig Schläfli (1850) gave a geometric argument that

$$R_n(d) = \begin{cases} 1 & d = 0, n \geq 1, \\ 1 & n = 0, d \geq 1, \\ R_{n-1}(d) + R_{n-1}(d-1) & n \geq 1, d \geq 1. \end{cases}$$

Use Schläfli's recurrence and induction on n to prove that

$$R_n(d) = \binom{n}{d} + \binom{n}{d-1} + \cdots + \binom{n}{1} + \binom{n}{0} \text{ for all } n \geq 0, d \geq 0.$$

Hint: For all $n \geq 0$, consider the statement

$$P(n) = \text{“}R_n(d) = \binom{n}{d} + \binom{n}{d-1} + \cdots + \binom{n}{1} + \binom{n}{0} \text{ for all } d \geq 0\text{”}.$$

Check that $P(0)$ is true. Then fix some arbitrary $n \geq 1$ and assume for induction that $P(n)$ is true. In this case, prove that $P(n+1)$ is also true. You will need to use the recurrence formula for Pascal's Triangle.

Proof. By definition we have $R_0(d) = 1$ for any $d \geq 0$. Also by definition, we have $\binom{0}{k} = 0$ for any $k \neq 0$. Thus for any $d \geq 0$ we have

$$\binom{0}{d} + \binom{0}{d-1} + \cdots + \binom{0}{0} = 0 + 0 + \cdots + 0 + 1 = 1 = R_0(d).$$

This shows that the statement $P(0)$ is true.

[2]A hyperplane is a flat $(d-1)$-dimensional shape in d-dimensional space. Never mind.

Now **fix** some arbitrary $n \geq 0$ and **assume** for induction that $P(n)$ is true. That is, suppose that for any $d \geq 0$ we have

$$R_n(d) = \binom{n}{d} + \binom{n}{d-1} + \cdots + \binom{n}{0}.$$

In this case we will prove for any $d \geq 0$ that

$$R_{n+1}(d) = \binom{n+1}{d} + \binom{n+1}{d-1} + \cdots + \binom{n+1}{0}.$$

How? From the definition of $R_n(d)$ and the induction hypothesis, we have

$$\begin{aligned} R_{n+1}(d) &= R_n(d) + R_n(d-1) \\ &= \binom{n}{d} + \binom{n}{d-1} + \cdots + \binom{n}{0} + \binom{n}{d-1} + \binom{n}{d-2} + \cdots + \binom{n}{0}. \end{aligned}$$

Now we group these terms in pairs and use the definition of Pascal's Triangle. Only one of the terms doesn't get paired up:

$$\begin{aligned} R_{n+1}(d) &= \binom{n}{d} + \binom{n}{d-1} + \cdots + \binom{n}{0} + \binom{n}{d-1} + \binom{n}{d-2} + \cdots + \binom{n}{0} \\ &= \left[\binom{n}{d} + \binom{n}{d-1}\right] + \left[\binom{n}{d-1} + \binom{n}{d-2}\right] + \cdots + \left[\binom{n}{1} + \binom{n}{0}\right] + \binom{n}{0} \\ &= \binom{n+1}{d} + \binom{n+1}{d-1} + \cdots + \binom{n+1}{1} + \binom{n}{0}. \end{aligned}$$

But it's okay because $\binom{n}{0} = \binom{n+1}{0} = 1$. Hence we have

$$R_{n+1}(d) = \binom{n+1}{d} + \binom{n+1}{d-1} + \cdots + \binom{n+1}{0},$$

as desired. □

Remark: The Steiner-Schläfli Theorem was pure recreational mathematics. But recreational mathematics has a habit of becoming useful. See Gilbert Strang's *Linear Algebra and Learning from Data* (page 381) for an application to neural networks.