# Math 309 Fall 2022
# Homework 1 Drew Armstrong

We will use the following notations for sums of (all, even, odd) $p$th powers:

$$S_p(n) = 1^p + 2^p + \cdots + n^p = \sum_{k=1}^{n} k^p,$$

$$SE_p(n) = 2^p + 4^p + 6^p + \cdots + (2n)^p = \sum_{k=1}^{n} (2k)^p,$$

$$SO_p(n) = 1^p + 3^p + 5^p + \cdots + (2n-1)^p = \sum_{k=1}^{n} (2k-1)^p.$$

**1.** Consider the following statement $P(n) =$ "$S_3(n) = n^2(n+1)^2/4$". In this problem you will prove by induction that $P(n)$ is true for all integers $n \geq 1$.

(a) Check by hand that $P(n)$ is true for $n = 1, 2, 3, 4$.

(b) Now fix some arbitrary $n \geq 1$ and **assume for induction** that $P(n)$ is a true statement. In this case, prove that $P(n+1)$ is also a true statement. [Hint: Use the recurrence $S_3(n+1) = S_3(n) + (n+1)^3$.]

**2.** Find explicit formulas for $SE_2(n)$ and $SO_2(n)$. [Hint: You may assume that $\sum_{k=1}^{n} k = n(n+1)/2$ and $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$.]

**3.** Define the sequence $C_0, C_1, C_2, C_3 \ldots$ by the following initial condition and recurrence:

$$C_n := \begin{cases} 1 & \text{if } n = 0, \\ C_{n-1} + n^2 - n & \text{if } n \geq 1. \end{cases}$$

Find a closed formula for $C_n$.

**4.** The sequence of *factorials* $0!, 1!, 2!, \ldots$ is defined as follows:

$$n! := \begin{cases} 1 & \text{if } n = 0, \\ (n-1)! \cdot n & \text{if } n \geq 1. \end{cases}$$

You will prove by induction that $n! > 3^n$ for all $n \geq 7$.

(a) Verify that $7! > 3^7$.

(b) Now fix some arbitrary $n \geq 7$ and assume for induction that $n! > 3^n$. In this case, prove that $(n+1)! > 3^{n+1}$. [Hint: Use the facts $(n+1)! = n! \cdot (n+1)$ and $n + 1 > 3$.]

Recall the definition of Pascal's Triangle. For all integers $n, k$ with $n \geq 0$ we have

$$\binom{n}{k} := \begin{cases} 1 & n = 0, k = 0, \\ 0 & n = 0, k \neq 0, \\ \binom{n-1}{k-1} + \binom{n-1}{k} & n \geq 1, k = \text{anything}. \end{cases}$$

This definition implies that $\binom{n}{k} = 0$ for $k < 0$ or $k > n$ and $\binom{n}{k} = 1$ for $k = 0$ or $k = n$. The Binomial Theorem says that for all numbers $x$ we have

$$(1+x)^n = \sum_{k} \binom{n}{k} x^k.$$

**5.** Use the Binomial Theorem to prove the following identity for all $n \geq 1$:

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n \cdot 2^{n-1}.$$

[Hint: Differentiate with respect to $x$.]

**6.** Let $R_n(d)$ be the maximum number of $d$-dimensional regions formed by $n$ hyperplanes in $d$-dimensional space.[1] Ludwig Schläfli (1850) gave a geometric argument that

$$R_n(d) = \begin{cases} 1 & d = 0, n \geq 1, \\ 1 & n = 0, d \geq 1, \\ R_{n-1}(d) + R_{n-1}(d-1) & n \geq 1, d \geq 1. \end{cases}$$

Use Schläfli's recurrence and induction on $n$ to prove that

$$R_n(d) = \binom{n}{d} + \binom{n}{d-1} + \cdots + \binom{n}{1} + \binom{n}{0} \text{ for all } n \geq 0, d \geq 0.$$

Hint: For all $n \geq 0$, consider the statement

$$P(n) = \text{“}R_n(d) = \binom{n}{d} + \binom{n}{d-1} + \cdots + \binom{n}{1} + \binom{n}{0} \text{ for all } d \geq 0\text{”}.$$

Check that $P(0)$ is true. Then fix some arbitrary $n \geq 1$ and assume for induction that $P(n)$ is true. In this case, prove that $P(n+1)$ is also true. You will need to use the recurrence formula for Pascal's Triangle.

[1] A hyperplane is a flat $(d-1)$-dimensional shape in $d$-dimensional space. Never mind.