Math 309 **Exam 3**
Fall 2022 **Tues Dec 6**

No electronic devices are allowed. There are 5 page and 5 problems. Each problem is worth 6 points, for a total of 30 points.

1. Complete Graphs. Let K_n be the complete graph on n vertices. Let $K_{m,n}$ be the complete bipartite graph on $m+n$ vertices.

(a) How many edges does K_7 have?

Solution. In general, K_n has $\binom{n}{2}$ edges. So K_7 has $\binom{7}{2} = 21$ edges.

(b) How many edges does $K_{4,5}$ have?

Solution. In general, $K_{m,n}$ has mn edges. So $K_{4,5}$ has $4 \cdot 5 = 20$ edges.

(c) How many edges are in the complement of $K_{4,5}$? [The complement has the same vertices as $K_{4,5}$, but switches edges with non-edges.]

Solution. In general, the complement of $K_{m,n}$ has $\binom{m}{2} + \binom{n}{2}$ edges, so the complement of $K_{4,5}$ has $\binom{4}{2} + \binom{5}{2} = 6 + 10 = 16$ edges.

Remark: Given a simple graph G with n vertices, with complement G'. Then

$$\#(\text{edges in } G) + \#(\text{edges in } G') = \binom{n}{2},$$

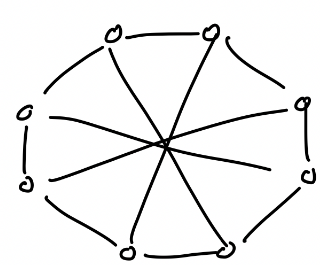
because each of the possible $\binom{n}{2}$ edges between the n vertices occurs in exactly one of G or G'. This formula agrees with our solutions to (b) and (c) because

$$\begin{aligned}\#(\text{edges in } K_{4,5}) + \#(\text{edges in } K'_{4,5}) &= 20 + 16 \\ &= 36 \\ &= \binom{4+5}{2}.\end{aligned}$$

2. Regular Graphs. A graph is called *d-regular* when every vertex has degree d.

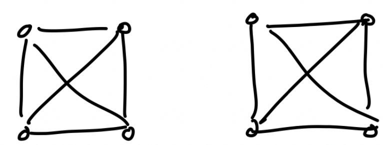
(a) Draw a connected 3-regular graph with 8 vertices.

Solution.



(b) Draw a non-connected 3-regular graph with 8 vertices.

Solution.



(c) Explain why a 3-regular graph with 9 vertices does not exist. [Hint: Handshaking.]

Solution. The Handshaking Lemma says that the sum of the vertex degrees equals twice the number of edges. If there existed a 3-regular graph with 9 vertices and e edges then we would have

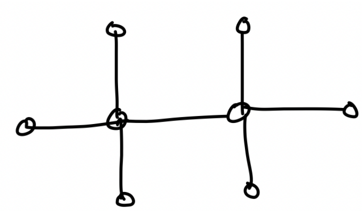
$$2e = \sum \text{vertex degrees} = \underbrace{3 + 3 + \cdots + 3}_{9 \text{ times}} = 3 \cdot 9 = 27,$$

which is impossible because $2e$ is even.

3. Trees. A *tree* is a connected graph with $e = n - 1$, where n is the number of vertices and e is the number of edges.

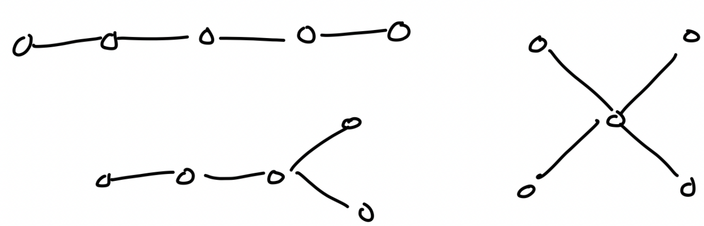
(a) Draw a tree with vertex degrees $1, 1, 1, 1, 1, 1, 4, 4$.

Solution.



(b) Draw three non-isomorphic trees, each with 5 vertices.

Solution.



(c) Explain why there is no tree with vertex degrees $1, 1, 1, 1, 2$.

Solutions. A tree with vertex degrees $1, 1, 1, 1, 2$ would have $n = 5$ vertices and $e = n - 1 = 4$ edges. On the other hand, by the Handshaking Lemma we must have

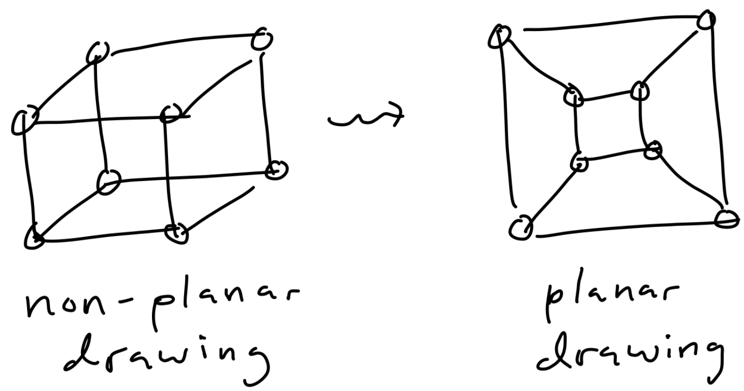
$$2e = 1 + 1 + 1 + 1 + 2 = 6,$$

which implies that $e = 3$. Contradiction.

4. Planar Graphs. A graph is called *planar* if it can be drawn in the plane with no crossing edges. Such a drawing divides the plane into *faces*. The *degree* of a face is the number of edges along its perimeter.

(a) Make a planar drawing of a graph with six faces, each of degree 4. [Hint: A cube.]

Solution.



(b) Let G be a planar graph drawing with n vertices, e edges and f faces. Suppose that every face has degree 4. In this case explain why $2e = 4f$.

Solution. The Handshaking Lemma for faces says that

$$2e = \sum \text{face degrees} = \underbrace{4 + 4 + \cdots + 4}_{f \text{ times}} = 4f.$$

(c) Continuing from part (b), use Euler's formula $n - e + f = 2$ to show that $e = 2n - 4$. Check that your drawing in part (a) satisfies this equation.

Solution. We have

$$\begin{aligned} n - e + f &= 2 \\ n - e + e/2 &= 2 & f = e/2 \text{ from (b)} \\ 2n - 2e + e &= 4 \\ 2n - 4 &= e. \end{aligned}$$

This agrees with our drawing in part (a) which has $n = 8$ vertices and $e = 12$ edges.

5. Induction. Let G be a graph with n vertices, e edges and k connected components. In this problem you will prove by induction on e that $n - k \leq e$.

(a) If $e = 0$, explain why we must have $n - k \leq e$.

Solution. If $e = 0$ then our graph consists of n disconnected vertices, and hence $k = n$ connected components. Hence $n - k = n - n = 0 \leq e$.

(b) Now suppose that $e \geq 1$ and let G' be a graph obtained from G by deleting a random edge. (Don't delete any vertices.) Let n', e', k' be the numbers of vertices, edges and components of the graph G'. Express n', e', k' in terms of n, e, k.

Solution. Let G' be obtained from G by deleting an arbitrary edge. Since we deleted a single edge we have $e' = e - 1$. But we didn't delete any vertices, so $n' = n$. What about connected components? Deleting an edge might increase the connected components by one: $k' = k + 1$. Of it might not change the number of connected components: $k' = k$.

(c) By induction we may suppose that $n' - k' \leq e'$. Combine this with your answer from part (b) to prove that $n - k \leq e$.

Solution. Suppose for induction that $n' - k' \leq e'$ in G'. In this case we will show that $n - k \leq e$ in the original graph G. From part (b) we have $n' = n$ and $e' = e - 1$. There are two cases for k':

- If $k' = k + 1$ then we have
$$n - k = n' - (k' - 1) = n' - k' + 1 \leq e' + 1 = e.$$
- If $k' = k$ then we have
$$n - k = n' - k' \leq e' = e - 1 \leq e.$$

In either case, we have $n - k \leq e$ as desired. □