Math 309 **Summer 2019**
Quiz 5 **Drew Armstrong**

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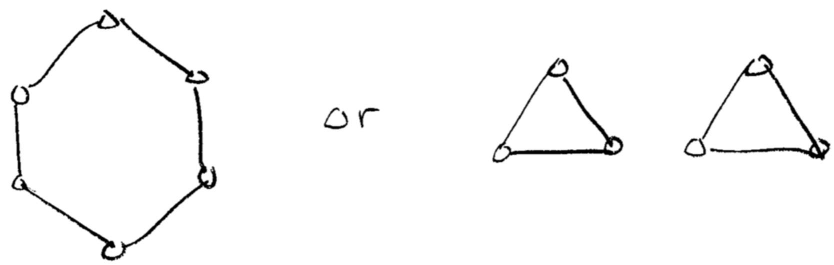
1. Degrees. Every graph in this problem has 6 vertices.

(a) Draw a graph with degrees $2, 2, 2, 2, 2, 2$.



(b) Draw a graph with degrees $1, 1, 2, 2, 2, 2$.

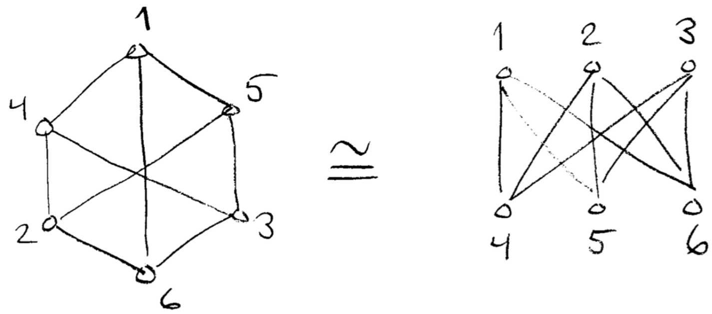
(c) Explain why there is no graph with degrees $1, 1, 1, 2, 2, 2$.

Proof. The degree sum of a graph is always even (because it equals twice the number of edges), but $1 + 1 + 1 + 2 + 2 + 2 = 9$ is an odd number. $\square$

2. Isomorphism.

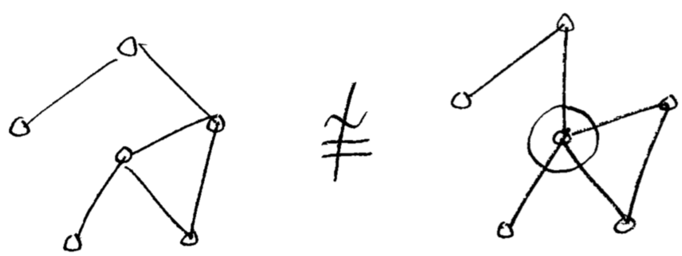
(a) Prove that the following graphs are isomorphic.

Observe that the labelings match:



(b) Prove that the following graphs are **not** isomorphic.

The degrees are not the same. For example the right graph has a vertex of degree 4 but the left graph does not:



(c) Draw **two non-isomorphic trees**, each with 4 vertices.

Here they are:

5. Graphs.

(a) Draw a disconnected 2-regular graph with 7 vertices.

(b) Draw a graph on 7 vertices with degree sequence $1, 2, 2, 2, 2, 2, 3$.

(c) Let $G = K_{4,4}$ be the complete bipartite graph on two sets of 4 vertices. Tell me the number of edges in G and the number of edges in the complement $\overline{G}$.

where the multinomial coefficients are defined by

$$\binom{\ell}{k_1, k_2, \ldots, k_n} = \frac{\ell!}{k_1! k_2! \cdots k_n!}$$

and where the sum is taken over all $k_1, \ldots, k_n \in \mathbb{N}$ such that $k_1 + \cdots k_n = \ell$.

- Substituting $a_1 = \cdots = a_n = 1$ into the multinomial theorem gives

$$n^\ell = \sum \binom{\ell}{k_1, \ldots, k_n}.$$

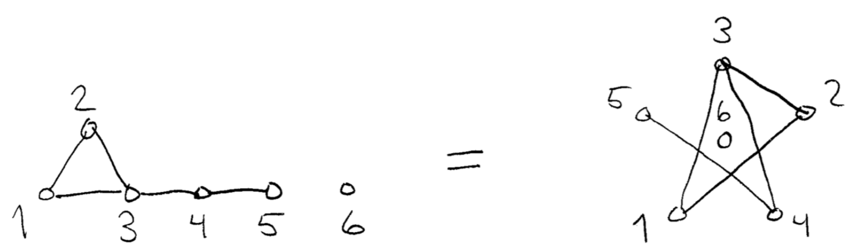
What does this mean? The left side counts the words of length ℓ from the alphabet $\{a_1, \ldots, a_n\}$. The right side counts the same words, but it groups them according to the number of each type of letter. We use the fact that

$$\binom{\ell}{k_1, k_2, \ldots, k_n} = \#\left\{ \begin{array}{c} \text{words of length } \ell \text{ containing} \\ k_i \text{ copies of } a_i \text{ for each } i \end{array} \right\}.$$

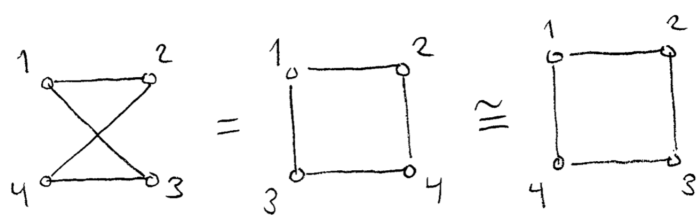
- Example: How many arrangements of the letters $e, f, f, l, o, r, e, s, c, e, n, c, e$?

Topics from Chapter 5

- A simple graph is a set of vertices, together with a set of unordered pairs of vertices, called edges. For example, let $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{3, 4\}, \{4, 5\}\}$.
- It is helpful to draw a graph, but the way you draw it is not important:



- If you permute labels (or if you don't draw labels) then you obtain *isomorphic graphs*:



- To prove that two graphs are isomorphic you must label them. To prove that two graphs are **not** isomorphic you need a trick.
- The easiest trick is to look at the degrees, since these are preserved under isomorphism. Let $G = (V, E)$ be a simple graph. Then for each vertex $u \in V$ we define its degree as

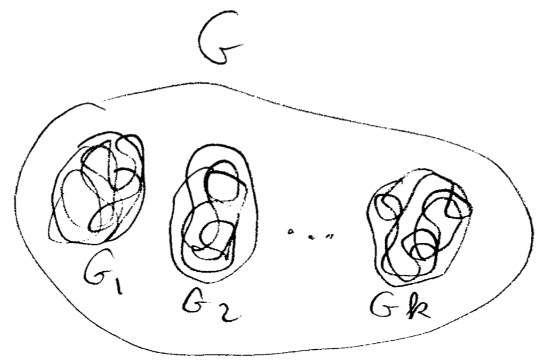
$$\deg(u) := \#\{v \in V : \{u, v\} \in E\}.$$

- The Handshaking Lemma says that

$$\sum_{u \in V} \deg(u) = 2 \cdot \#E.$$

  Proof: Let L be the set of lollipops in the graph (a lollipop is an edge together with one of its vertices). By choosing the edge first we have $\#L = 2 \cdot \#E$. By choosing the vertex first we have $\#L = \sum_{u \in V} \deg(u)$. □
- It follows that the number of odd-degree vertices is even. For example, there is no graph with degree sequence $2, 2, 2, 3, 3, 4, 5$ because $2 + 2 + 2 + 3 + 3 + 4 + 5$ is an odd number.
- A graph is called d-regular if each vertex has degree d. If G is a d-regular graph with n vertices then it follows from the First Theorem that dn is even. For example, there does not exist a 3-regular graph on 7 vertices. Exercise: Draw a 3 regular graph on 8 vertices. Exercise: Prove that there exist two non-isomorphic 3-regular graphs on 6 vertices.
- Example: The hypercube Q_n is an n-regular graph on 2^n vertices. The vertices are binary strings of length n and the edges are "bit flips." Exercise: Compute the number of edges.[1]
- Famous graphs include the path P_n, cycle C_n, complete graph K_n and the complete bipartite graph $K_{m,n}$. You should know all the important properties of these graphs and be able to draw them.
- Let $G = (V, E)$ be a simple graph. The complement $\overline{G}$ has the same vertices but the edges and the non-edges have been flipped. Thus if G has n vertices and e edges then $\overline{G}$ has n vertices and $\binom{n}{2} - e$ edges. Exercise: Draw the graph $K_{3,4}$ and its complement.
- A u, v-walk of length ℓ in $G = (V, E)$ is a sequence of vertices $u = v_0, v_1, \ldots, v_\ell = v \in V$ such that $\{v_{i-1}, v_i\} \in E$ for all $i \in \{1, \ldots, \ell\}$. A path is a walk with no repeated vertex. By recursion every u, v-walk contains a u, v-path. Proof: Find a repeated vertex and cut out everything in between. Repeat until there is no repeated vertex.
- We say that the graph is connected if for all $u, v \in V$ there exists a u, v-path. More generally, we define the connected components $G = G_1 \cup G_2 \cup \cdots \cup G_k$ so that vertices $u, v \in V$ are connected if and only if they are in the same component. Picture:

[1] Hao Huang recently (July 1st, 2019) proved the following result, which settled a 30-year-old conjecture: Let A be a subset of vertices in the hypercube Q_n satisfying $\#A \geqslant 2^{n-1} + 1$. Then there exists a vertex $a \in A$ that has at least $\sqrt{n}$ neighbors in A.



- If G has n vertices, e edges and k components then $n - k \leqslant e$. [Remark: This result holds even for multigraphs.]

  Proof by induction on e: Fix $n \geqslant 0$. If $e = 0$ then $k = n$ and hence $n - k = 0 = e$. Now suppose that $e \geqslant 1$ and delete a random edge to obtain a graph G' with n', e', k'. Note that $n' = n$ and $e' = e - 1$. Since $e' < e$ we can assume by induction that $n' - k' \leqslant e'$. But we also know that $k' \leqslant k + 1$ since deleting an edge creates at most one extra component (and maybe none). Hence

  $$e = e' - 1 \geqslant (n' - k') - 1 = n - 1 - k' \geqslant n - 1 - (k + 1) = n - k.$$

  □

- If G is a simple graph with n vertices, e edges and k components then $e \leqslant \binom{n-(k-1)}{2}$. You do not need to prove this. The number of edges is maximized when every component but one is a single vertex and the last component is a complete graph on $n - (k - 1)$ vertices.

- A circuit is a walk that begins and ends at the same vertex. A cycle is a circuit that has no repeated vertices (except for the basepoint). Every circuit contains a cycle.

- First Application: A graph is called bipartite if it has **no odd cycles**. Equivalently, we can color the vertices with two colors such that no two vertices of the same color share an edge. (You don't need to know the proof.)

- Second Application: A graph is called a forest if it has **no cycles at all**. One can show that this happens exactly when $e = n - k$, i.e., when the number of edges is minimized. A forest with one connected component ($k = 1$) is called a tree. In other words, a tree is a connected graph with no cycles. Equivalently, a tree is a connected graph on n vertices with $e = n - 1$ edges. Exercise: Draw a forest with $n = 12$ and $k = 3$. Verify that the number of edges is $e = n - k = 9$.

- Let G be a tree on vertex set $\{1, 2, \ldots, n\}$ and let $d_i := \deg(i)$. Since G has $e = n - 1$ edges we must have

  $$\sum_{i=1}^{n} d_i = 2(n - 1)$$

and hence

$$\sum_{i=1}^{n}(d_i - 1) = \sum_{i=1}^{n} d_i - \sum_{i=1}^{n} 1 = 2(n-1) - n = 2n - 2 - n = n - 2.$$

- Cayley's Tree Formula says that

$$\binom{n-2}{d_1 - 1, d_2 - 1, \ldots, d_n - 1} = \#\left\{ \begin{array}{c} \text{trees on vertex set } \{1, \ldots, n\} \\ \text{where vertex } i \text{ has degree } d_i \end{array} \right\}.$$

  By summing over all possible degrees we obtain

$$\#\{\text{labeled trees on } n \text{ vertices}\} = \sum \binom{n-2}{d_1 - 1, d_2 - 1, \ldots, d_n - 1} = n^{n-2}.$$

  Exercise: Verify that this last step follows from the multinomial theorem.

- Prüfer's proof of Cayley's Formula: Given a tree T on $\{1, 2, \ldots, n\}$, delete the smallest leaf (vertex of degree one) and let p_1 be the name of its parent. Repeat to obtain a sequence $(p_1, p_2, \ldots, p_{n-2})$ called the *Prüfer code* of the tree. One can show that every word of length $n-2$ from the alphabet $\{1, \ldots, n\}$ is the Prüfer code of some tree. (You don't need to show this.) Furthermore, the number i shows up exactly $d_i - 1$ times in the code. Example: