Math 309
Exam 3 Practice Problems with Solutions

Problem 1. Let G be a connected, simple graph (no loops, no multiple edges) with n vertices and e edges. If G has a planar drawing then we know that $e \leq 3n - 6$. (You don't need to prove this.) Use this fact to prove that the complete graph K_6 has no planar drawing.

Solution. The graph K_6 has $n = 6$ vertices and $e = \binom{6}{2} = 15$ edges. But then $e \leq 3n - 6$ is false because $15 \leq 18 - 6$ is false. Hence K_6 has no planar drawing.

Remark. In general, the complete graph K_n has n vertices and $\binom{n}{2}$ edges, because each pair of vertices is connected by an edge.

Problem 2. Let G be a connected, simple graph (no loops, no multiple edges) with n vertices and e edges. Suppose that G has a planar drawing with f faces. In this case show that $2e \geq 3f$.

Solution. By the Handshaking Lemma we have

$$2e = \sum \text{face degrees.}$$

Since G has no loops and no multiple edges, each face in the drawing must have degree ≥ 3, hence

$$2e = \sum \text{face degrees} \geq \underbrace{3 + 3 + \cdots + 3}_{f \text{ times}} = 3f.$$

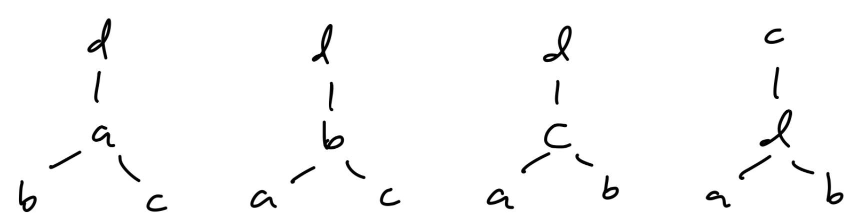
Problem 3. A tree is a connected graph with no cycles. Every tree can be drawn in the plane. (You don't need to prove this.) If T is a tree with n vertices and e edges, use Euler's formula to show that $e = n - 1$.

Solution. If a graph with n vertices and e edges has a planar drawing with f faces then Euler's formula says $n - e + f = 2$. A planar drawing of a tree has exactly one face, because it has no cycles. Hence Euler's formula gives

$$\begin{aligned} n - e + f &= 2 \\ n - e + 1 &= 2 \\ n - 1 &= e. \end{aligned}$$

Problem 4. There are exactly 4 graphs with vertices $\{a, b, c, d\}$ and vertex degrees $1, 1, 1, 3$. Draw them.

Solution.



Problem 5. For any graph with n vertices, e edges and k connected components, we must have $n - k \leq e$. Use this to show that a graph with vertex degrees $1, 1, 1, 1, 1, 1, 2, 2, 2$ must have at least three connected components.

Solution. A graph with these degrees has $n = 9$ and

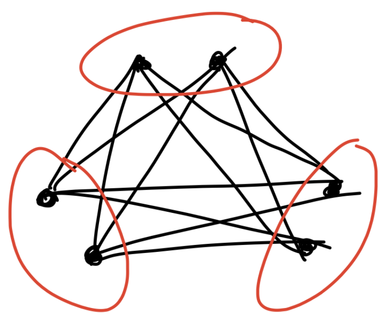
$$e = \frac{1}{2}(1 + 1 + 1 + 1 + 1 + 1 + 2 + 2 + 2) = 6.$$

But then $n - 1 \leq e$ and $n - 2 \leq e$ are false, so this graph cannot have one or two components. It must have at least three connected components.

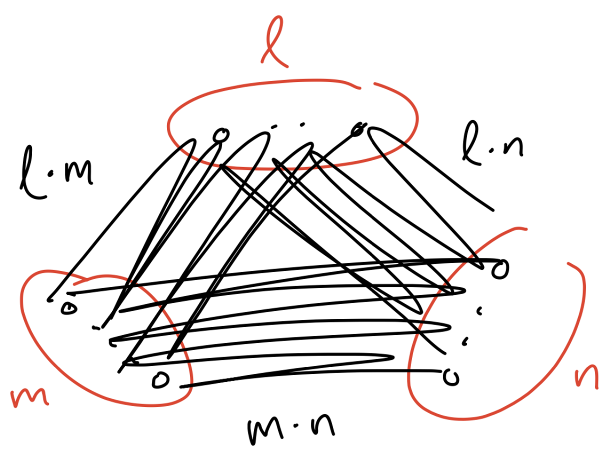
Problem 6. We define the graph $K_{\ell,m,n}$ as follows. There are $\ell + m + n$ vertices, divided into three sets A, B, C of size ℓ, m, n, respectively. The edges consist of all pairs ab, ac and bc with $a \in A$, $b \in B$ and $c \in C$.

(a) Draw the graph $K_{2,2,2}$. How many edges does it have?
(b) How many edges are in the graph $K_{\ell,m,n}$?

Solution. (a):



(b): There are ℓm edges between A and B, there are ℓn edges between A and C, and there are mn edges between B and C:



Hence the total number of edges is

$$\ell m + \ell n + mn.$$