Math 309 **Exam 2**
Fall 2022 **Thurs Nov 3**

No electronic devices are allowed. There are 5 page and 5 problems. Each problem is worth 6 points, for a total of 30 points.

1. Arithmetic.

(a) Compute the quotient and remainder of 35 modulo 14.

The quotient q and remainder r are the unique integers satisfying

$$\begin{cases} 35 = 14q + r, \\ 0 \leq r < 14. \end{cases}$$

Note that $q = 2$ and $r = 7$ will work.

(b) Convert the binary expression $(1111)_2$ into decimal (i.e., base 10).

$$\begin{aligned} (1111)_2 &= 1 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3 \\ &= 1 + 2 + 4 + 8 \\ &= 15. \end{aligned}$$

(c) Convert the decimal expression 17 into base 3.

Divide 17 by 3 and then divide each successive quotient by 3:

$$\begin{aligned} 17 &= 3 \cdot 5 + 2 \\ 5 &= 3 \cdot 1 + 2 \\ 1 &= 3 \cdot 0 + 1. \end{aligned}$$

Then use back substitution:

$$\begin{aligned} 17 &= 3 \cdot 5 + 2 \\ &= 3 \cdot (3 \cdot 1) + 2 \\ &= 3 \cdot (3 \cdot (3 \cdot 0 + 1) + 2) + 2 \\ &= 3^3 \cdot 0 + 3^2 \cdot 1 + 3^1 \cdot 2 + 2. \end{aligned}$$

We conclude that

$$17 = (0122)_3 = (122)_3.$$

2. Modular Arithmetic.

(a) Apply the Extended Euclidean Algorithm to find integers $x, y \in \mathbb{Z}$ satisfying

$$17x + 7y = 1.$$

We consider the set of triples (x, y, r) satisfying $17x + 7y = r$. Starting with the obvious triples $(1, 0, 17)$ and $(0, 1, 7)$ we perform the steps of the Euclidean Algorithm on the third coordinates until we obtain a triple of the form $(x, y, 1)$:

x	y	r	operation
1	0	17	
0	1	7	
1	−2	3	(row 1)−2·(row 2)
−2	5	1	(row 2)−2·(row 3)

We conclude that $17(-2) + 7(5) = 1$

(b) Use your answer from (a) to find a number $k \in \mathbb{Z}$ satisfying $7k \equiv 1 \pmod{17}$.

Since $17 \equiv 0 \pmod{17}$ we have

$$17 \cdot 5 \equiv 1 - 17(-2) \equiv 1 - 0(-2) \equiv 1 \pmod{17}.$$

(c) Use your answer from (b) to solve the congruence $7c \equiv 4 \pmod{17}$ for c.

We solve this by multiplying both sides by 5:

$$\begin{aligned} 7c &\equiv 4 \\ 5 \cdot 7c &\equiv 5 \cdot 4 \\ 1c &\equiv 20 \\ c &\equiv 3 \pmod{17}. \end{aligned}$$

3. Bijection.

(a) Write down all rearrangements of the letters a, a, b, b, b.

aabbb	*baabb*	*bbaab*	*bbbaa*
ababb	*babab*	*bbaba*	
abbab	*babba*		
abbba			

(b) Write down all subsets of size 2 from the set $\{1, 2, 3, 4, 5\}$.

$\{1, 2\}$	$\{2, 3\}$	$\{3, 4\}$	$\{4, 5\}$
$\{1, 3\}$	$\{2, 4\}$	$\{3, 5\}$	
$\{1, 4\}$	$\{2, 5\}$		
$\{1, 5\}$			

(c) Describe an explicit bijection between your answers from (a) and (b).

The word with a's in positions i and j corresponds to the subset $\{i,j\} \subseteq \{1,2,3,4,5\}$. I have elements in (a) and (b) to illustrate this bijection.

4. Counting. Count the possibilities in each case. Don't write out all the examples.

(a) Words of length 3 from the alphabet $\{a,b,c\}$. Repeated letters are allowed.

There are three possibilities for each letter, hence the total is

$$\underbrace{3}_{\text{1st letter}} \times \underbrace{3}_{\text{2nd letter}} \times \underbrace{3}_{\text{3rd letter}} = 3^3 = 27.$$

(b) Rearrangements of the letters b,u,b,b,l,e.

There are 3 b's, 1 u, 1 l and 1 e. Hence the total number of arrangements is

$$\frac{6!}{3!1!1!1} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1} = 120.$$

(c) Non-negative integer solutions to the equation $w + x + y + z = 5$.

Each solution corresponds to a sequence of 5 stars and 3 bars:

$$\underbrace{* \cdots *}_{w} | \underbrace{* \cdots *}_{x} | \underbrace{* \cdots *}_{y} | \underbrace{* \cdots *}_{z}$$

The number of such sequences is

$$\binom{8}{5,3} = \frac{8!}{5!3!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56.$$

5. Counting Proof. Let i,j,k be non-negative integers and suppose that $i + j + k = n$. Then we have the following identity:

$$\binom{n}{i}\binom{n-i}{j} = \frac{n!}{i!j!k!}.$$

(a) *Algebraic Proof.* Prove the identity using the formula $\binom{\ell}{m} = \frac{\ell!}{m!(\ell-m)!}$.

Since $n - i - j = k$ we have

$$\binom{n}{i}\binom{n-i}{j} = \frac{n!}{i!\cancel{(n-i)!}} \cdot \frac{\cancel{(n-i)!}}{j!(n-i-j)!} = \frac{n!}{i!j!(n-i-j)!} = \frac{n!}{i!j!k!}.$$

(b) *Counting Proof.* We know that $n!/(i!j!k!)$ is the number of words of length n containing i copies of "a", j copies of "b" and k copies of "c". Explain why these words are also counted by the formula $\binom{n}{i}\binom{n-i}{j}$.

In order to create a word of length n containing i copies of a, j copies of b and k copies of c, we first choose i positions from the set $\{1, \ldots, n\}$ and place the a's in these positions. Then we choose j positions from the remaining $n - i$ positions, and place the b's in these positions. At this point there are $n - i - j = k$ positions remaining, and we fill these positions with c's. The total number of possibilities is

$$\underbrace{\binom{n}{i}}_{\substack{\text{first choose where}\\ \text{to put the } a\text{'s}}} \times \underbrace{\binom{n-i}{j}}_{\substack{\text{then choose where}\\ \text{to put the } b\text{'s}}} .$$

We can also write

$$\underbrace{\binom{n}{i}}_{\substack{\text{first choose where}\\ \text{to put the } a\text{'s}}} \times \underbrace{\binom{n-i}{j}}_{\substack{\text{then choose where}\\ \text{to put the } b\text{'s}}} \times \underbrace{\binom{n-i-j}{k}}_{\substack{\text{then choose where}\\ \text{to put the } c\text{'s}}} ,$$

where the third factor equals 1 because $n - i - j = k$.

For example, with $n = 7$, $i = 3$, $j = 2$ and $k = 2$, we first choose a set of size 3 from $\{1, 2, 3, 4, 5, 6, 7\}$. Say we choose $\{2, 3, 7\}$. Then we choose a set of size 2 from the remaining positions $\{1, 4, 5, 6\}$. Say we choose $\{4, 6\}$. The resulting word is

$$\underbrace{c}_{1}\ \underbrace{a}_{2}\ \underbrace{a}_{3}\ \underbrace{b}_{4}\ \underbrace{c}_{5}\ \underbrace{b}_{6}\ \underbrace{a}_{7}$$