Math 309 **Fall 2014**
Exam 2 (Wed Oct 29) **Drew Armstrong**

There are 5 problems, with a total of 20 parts. Each part is worth 2 points, for a total of 40 points. If two exams are submitted with identical answers then **both** will receive 0 points.

1. Boolean Algebra. Recall that the Boolean function $\Rightarrow$ is defined by

$$P \Rightarrow Q := (\neg P) \vee Q.$$

(a) Draw the truth table for $P \Rightarrow Q$.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

(b) Accurately state De Morgan's Law.

For all Boolean variables P and Q we have

- $\neg(P \vee Q) = (\neg P) \wedge (\neg Q)$
- $\neg(P \wedge Q) = (\neg P) \vee (\neg Q)$

(c) Use De Morgan's Law to prove that $(P \Rightarrow Q) = ((\neg Q) \Rightarrow (\neg P))$.

Oops. De Morgan's Law is not actually necessary to prove this. Sorry about that. We have

$$\begin{aligned}(\neg Q) \Rightarrow (\neg P) &= \neg(\neg Q) \vee (\neg P) \\ &= Q \vee (\neg P) \\ &= (\neg P) \vee Q \\ &= P \Rightarrow Q.\end{aligned}$$

(d) Use a truth table to prove that $(P \Rightarrow Q) \neq (Q \Rightarrow P)$.

Note that $P \Rightarrow Q$ and $Q \Rightarrow P$ differ in the second and third rows:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

2. Boolean Functions. A Boolean function with m inputs and n outputs has the form

$$\varphi : \{T, F\}^m \to \{T, F\}^n.$$

(a) Explicitly write down all the elements of the set $\{T,F\}^3$.

Recall that $\{T,F\}^3 = \{T,F\}\times\{T,F\}\times\{T,F\}$ consists of **ordered triples** of elements from $\{T,F\}$. The elements of the set are:

$$\begin{array}{ccc} & (T,T,T) & \\ (T,T,F) & (T,F,T) & (F,T,T) \\ (T,F,F) & (F,T,F) & (F,F,T) \\ & (F,F,F) & \end{array}$$

(b) How many elements does the set $\{T,F\}^n$ have?

$$\#(\{T,F\}^n) = (\#\{T,F\})^n = 2^n$$

(c) How many functions are there from $\{T,F\}^m$ to $\{T,F\}^n$?

The number of functions from $\{T,F,\}^m$ to $\{T,F\}^n$ is

$$\#(\{T,F\}^n)^{\#(\{T,F\}^m)} = (2^n)^{(2^m)}$$

(d) How many Boolean functions are there with 3 inputs and 1 output?

When $m = 3$ and $n = 1$ the number of functions is

$$(2^n)^{(2^m)} = (2^1)^{(2^3)} = 2^8 = 256.$$

3. Subsets $\leftrightarrow$ Binary Strings. Consider the set $U = \{1,2,3,4,5\}$.

(a) Make a table to display the number of subsets of U with size k, for $k = 0,1,2,3,4,5$.

k	0	1	2	3	4	5
# subsets of size k	1	5	10	10	5	1

(b) Explicitly write down all of the subsets of U containing **two elements**.

$$\begin{array}{llll} \{1,2\}, & \{1,3\}, & \{1,4\}, & \{1,5\}, \\ \{2,3\}, & \{2,4\}, & \{2,5\}, & \\ \{3,4\}, & \{3,5\}, & & \\ \{4,5\} & & & \end{array}$$

(c) Explicitly write down all of the binary strings with two "1"s and three "0"s.

$$\begin{array}{llll} 11000, & 10100, & 10010, & 10001, \\ 01100, & 01010, & 01001, & \\ 00110, & 00101, & & \\ 00011 & & & \end{array}$$

(d) Draw lines between your answers to (b) and (c) to demonstrate a natural bijection.

The bijection is implied by the way I drew the two sets.

4. The Binomial Theorem.

(a) Accurately state the Binomial Theorem.

For any number a and b, and for any integer $n \geq 0$ we have

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

(b) Use the Binomial Theorem to prove that $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$.

Subsitute $a = 1$ and $b = 1$.

(c) Use the Binomial Theorem to prove that $3^n = 1\binom{n}{0} + 2\binom{n}{1} + 4\binom{n}{2} + \cdots + 2^n\binom{n}{n}$.

Substitute $a = 2$ and $b = 1$.

(d) Use the Binomial Theorem to prove that $0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n}$.

Substitute $a = -1$ and $b = 1$.

5. Binomial Coefficients.

(a) State the formula for $\binom{n}{k}$.

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

(b) Use the formula to prove that $k\binom{n}{k} = n\binom{n-1}{k-1}$.

$$k\binom{n}{k} = k\frac{n!}{k!\,(n-k)!} = \frac{n!}{(k-1)!\,(n-k)!} = n\frac{(n-1)!}{(k-1)!\,(n-k)!} = n\binom{n-1}{k-1}$$

For parts (c) and (d), suppose you want to choose a committee of k people from a set of n people. One person on the committee will be called the "president".

(c) Explain why the number of ways to do this is $k\binom{n}{k}$.

If we choose the committee first and then the president, there are $\binom{n}{k}$ ways to choose the committee and then k ways to choose the president from the committee. Hence the total number of choices is $\binom{n}{k}k$.

(d) Explain why that the number of ways to do this is $n\binom{n-1}{k-1}$.

It we choose the president first and then the committee, there are n ways to choose the president and then $\binom{n-1}{k-1}$ ways to choose the other $k-1$ members of the committee from the remaining $n-1$ people. Hence the total number of choices is $n\binom{n-1}{k-1}$.

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Quiz 3 **Drew Armstrong**

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1. Accurately state the Binomial Theorem.

For all integers $n \geq 0$ and for all real numbers x, y we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \cdot x^k y^{n-k}$$

2. Draw Pascal's Triangle down the sixth row and use this to find the expansion of $(x+y)^6$.

Here is Pascal's Triangle:

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & & 1 & & & & & \\ & & & & 1 & & 2 & & 1 & & & & \\ & & & 1 & & 3 & & 3 & & 1 & & & \\ & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \end{array}$$

Therefore we have

$$(x+y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6.$$

3. Consider the set $S = \{1,2,3,4,5,6\}$.

(a) How many subsets does S have?

The number of subsets is $2^{\#S} = 2^6 = 64$. Alternatively, we can use Pascal's Triangle:

$$\sum_{k=0}^{6} \binom{6}{k} = 1 + 6 + 15 + 20 + 15 + 6 + 1 = 64.$$

(b) How many of these subsets contain an **even** number of elements? [Note: 0 is even.]

You may remember from class that the number of even subsets is $2^{\#S-1} = 2^5 = 32$. Alternatively, we can use Pascal's Triangle:

$$\binom{6}{0} + \binom{6}{2} + \binom{6}{4} + \binom{6}{6} = 1 + 15 + 15 + 1 = 32.$$

4.

(a) How many words can be made from k copies of "a" and $n-k$ copies of "b"?

This is the well-known binomial coefficient:

$$\frac{n!}{k!(n-k)!}.$$

(b) How many ways are there to arrange the letters "$t, e, n, n, e, s, s, e, e$" ?

There are 9 letters in total, in which

"t" appears 1 time,
"e" appears 4 times,
"n" appears 2 times, and
"s" appears 2 times.

Therefore the number of arrangements is the following multinomial coefficient:

$$\frac{9!}{1!4!2!2!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{2 \cdot 2} = 3,780.$$

Since there's extra white space, here's a free remark:

$$(t + e + n + s)^9 = \cdots + 3780 \cdot t^1 e^4 n^2 s^2 + \cdots .$$

1. Accurately state the Division Theorem.

For all integers $a, b \in \mathbb{Z}$ with $b \neq 0$, there exist unique integers $q, r \in \mathbb{Z}$ satisfying:

$$\begin{cases} a = qb + r, \\ 0 \leq r < |b|. \end{cases}$$

2. Let $a, b \in \mathbb{Z}$ and consider the following statement:

$$\text{“}2|a \Rightarrow 2|(ab).\text{”}$$

(a) Translate the statement into English.

“If a is even then ab is even.”
or
“If 2 divides a then 2 divides ab.”
or
“If there exists $k \in \mathbb{Z}$ such that $a = 2k$, then there exists $\ell \in \mathbb{Z}$ such that $ab = 2\ell$.”

(b) Prove that the statement is true.

Proof: If $2|a$ then by definition we have $a = 2k$ for some $k \in \mathbb{Z}$. But then we also have

$$ab = (2k)b = 2(kb),$$

which by definition says that $2|ab$. □

3. Apply the Euclidean Algorithm to compute greatest common divisor of 105 and 91.

$$\begin{aligned} \mathbf{105} &= 1 \cdot \mathbf{91} + \mathbf{14}, \\ \mathbf{91} &= 6 \cdot \mathbf{14} + \mathbf{7}, \\ \mathbf{14} &= 2 \cdot \mathbf{7} + \mathbf{0}. \end{aligned} \qquad \begin{aligned} \gcd(105, 91) &= \gcd(91, 14) \\ &= \gcd(14, 7) \\ &= \gcd(7, 0) = 7. \end{aligned}$$

4. Apply the Extended Euclidean Algorithm to find the **complete integer solution** $x, y \in \mathbb{Z}$ to the following linear equation:

$$8x + 5y = 1.$$

We make a table of triples $(x, y, z) \in \mathbb{Z}^3$ satisfying $8x + 5y = z$:

x	y	z
1	0	8
0	1	5
1	−1	3
−1	2	2
2	−3	1
−5	8	0

The second-last row gives us one particular solution:

$$8(2) + 5(-3) = 1.$$

And the last row gives us the complete homogeneous solution:

$$8(-5k) + 5(8k) = 0 \qquad \text{for all } k \in \mathbb{Z}.$$

Putting these together gives the complete solution:

$$8(2 - 5k) + 5(-3 + 8k) = 1 \qquad \text{for all } k \in \mathbb{Z}.$$

In other words:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} + k \begin{pmatrix} -5 \\ 8 \end{pmatrix} \qquad \text{for all } k \in \mathbb{Z}.$$

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1. Base b Arithmetic.

Convert the decimal number 111 into binary.

Set $q := 111$ and then repeatedly divide the quotient by 2:

$$\begin{aligned}
\mathbf{111} &= \mathbf{55}\cdot 2 & +1\\
\mathbf{55} &= \mathbf{27}\cdot 2 & +1\\
\mathbf{27} &= \mathbf{13}\cdot 2 & +1\\
\mathbf{13} &= \mathbf{6}\cdot 2 & +1\\
\mathbf{6} &= \mathbf{3}\cdot 2 & +0\\
\mathbf{3} &= \mathbf{1}\cdot 2 & +1\\
\mathbf{1} &= \mathbf{0}\cdot 2 & +1
\end{aligned}$$

We conclude that $111 = (1101111)_2$.

2. Induction Again. Fix some number $r \neq 1$.

Use induction to prove that $r^0 + r^1 + \cdots + r^n = (r^{n+1}-1)/(r-1)$ for all $n \geq 0$.

Proof. For the base case $n = 0$ we observe that

$$r^0 = \frac{r^1 - 1}{r-1} \quad \text{is a true statement.}$$

Now fix some integer $n \geq 0$ and assume for induction that

$$r^0 + r^1 + \cdots + r^n = \frac{r^{n+1}-1}{r-1} \quad \text{is a true statement.}$$

Then we also have

$$\begin{aligned}
r^0 + r^1 + \cdots + r^{n+1} &= r^0 + r^1 + \cdots + r^n + r^{n+1}\\
&= \frac{r^{n+1}-1}{r-1} + r^{n+1}\\
&= \frac{r^{n+1}-1}{r-1} + \frac{r^{n+1}(r-1)}{r-1}\\
&= \frac{r^{n+1}-1}{r-1} + \frac{r^{n+2}-r^{n+1}}{r-1}\\
&= \frac{r^{n+1}-1+r^{n+2}-r^{n+1}}{r-1}\\
&= \frac{r^{n+2}-1}{r-1}.
\end{aligned}$$

Hence the statement is true for $n+1$. □

2. Division With Remainder.

(a) Accurately State the Division Theorem.

For all integers $a, b \in \mathbb{Z}$ with $b > 0$, there exist unique integers $q, r \in \mathbb{Z}$ such that

$$\begin{cases} a = qb + r, \\ 0 \leq r < b. \end{cases}$$

(b) Use the Euclidean algorithm to compute $\gcd(100, 23)$.

First we divide 100 by 23 to get sone remainder r. Then we replace the pair $(100, 23)$ by $(23, r)$ and repeat:

$$\begin{aligned} \mathbf{100} &= 4 \cdot \mathbf{23} && +\mathbf{8} \\ \mathbf{23} &= 2 \cdot \mathbf{8} && +\mathbf{7} \\ \mathbf{8} &= 1 \cdot \mathbf{7} && +\mathbf{1} \\ \mathbf{7} &= 7 \cdot \mathbf{1} && +\mathbf{0} \end{aligned}$$

We conclude that $\gcd(100, 23) = 1$.

(c) Apply your work from (b) to find the continued fraction expansion of $100/23$.

The sequence of quotients $(4, 2, 1, 7)$ from part (b) tells us that

$$\frac{100}{23} = 4 + \frac{1}{2 + \frac{1}{1 + \frac{1}{7}}}.$$

Math 309 Quiz 4

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Drew Armstrong

1. Counting Words.

(a) Tell me the number of words of length 5 that can be made from the alphabet $\{a, b, c\}$.

$$\#(\text{words}) = \underbrace{3}_{\text{1st letter}} \times \underbrace{3}_{\text{2nd letter}} \times \underbrace{3}_{\text{3rd letter}} \times \underbrace{3}_{\text{4th letter}} \times \underbrace{3}_{\text{5th letter}} = 3^5 = 243.$$

(b) How many of the words from (a) contain 3 copies of a, 1 copy of b and 1 copy of c?

$$\binom{5}{3,1,1} = \frac{5!}{3!1!1!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 20.$$

(c) How many of the words from (a) contain 3 copies of a? [Hint: You need to add over all possible numbers of b's and c's.]

$$\binom{5}{3,2,0} + \binom{5}{3,1,1} + \binom{5}{3,0,2} = 10 + 20 + 10 = 40.$$

2. Algebraic vs Counting Proof. For all integers $n \geq 2$ we have the following identity:

$$n^2 = 2\binom{n}{2} + n.$$

(a) Give an algebraic proof of the identity.

Proof.

$$2\binom{n}{2} + n = 2\frac{n(n-1)}{2} + n = n(n-1) + n = (n^2 - n) + n = n^2.$$

□

(b) Give a counting proof of the identity. [Hint: Count words of length 2.]

Proof. Let W be the set of words of length 2 from an alphabet of size n. On the one hand we have

$$\#W = n^2.$$

On the other hand, let $A \subseteq W$ be the words with 2 different letters and let $B \subseteq W$ be the words with the same letter twice, so $\#W = \#A + \#B$. Then we have

$$\#A = \underbrace{\binom{n}{2}}_{\text{choose two letters}} \times \underbrace{2}_{\text{put them in order}} \quad \text{and} \quad \#B = \underbrace{n}_{\text{choose one letter}}.$$

□