**Math 309** **Fall 2014**
**Exam 1 (Wed Sept 24)** **Drew Armstrong**

There are 5 problems, each with 3 parts. Each part is worth 2 points, for a total of 30 points. If any two exams are submitted with identical answers then **both** exams will receive 0 points.

**1. Induction.** For all positive integers $n$ and $p$ we define

$$S_p(n) := 1^p + 2^p + 3^p + \cdots + n^p.$$

In this problem you will use induction to prove that $S_1(n) = n(n+1)/2$ for all $n \geq 1$.

(a) Check that the formula $S_1(n) = n(n+1)/2$ is true for $n = 1$, 2, and 3.

| $n$ | $S_1(n)$ | $n(n+1)/2$ | correct? |
|---|---|---|---|
| 1 | 1 | $\frac{1(1+1)}{2} = \frac{1\cdot 2}{2} = 1$ | ✓ |
| 2 | $1+2=3$ | $\frac{2(2+1)}{2} = \frac{2\cdot 3}{2} = 3$ | ✓ |
| 1 | $1+2+3=6$ | $\frac{3(3+1)}{2} = \frac{3\cdot 4}{2} = 6$ | ✓ |

(b) Find a simple equation relating $S_1(n)$ and $S_1(n+1)$.

$$S_1(n+1) = 1 + 2 + 3 + \cdots + n + (n+1)$$
$$S_1(n+1) = S_1(n) + (n+1).$$

(c) Prove that **if** $S_1(n) = n(n+1)/2$, **then** $S_1(n+1) = (n+1)(n+2)/2$. You must begin your proof with the words "Consider some $n \geq 1$ and assume that…".

Consider some $n \geq 1$ and assume that $S_1(n) = n(n+1)/2$. In this case we have

$$\begin{aligned} S_1(n+1) &= S_1(n) + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= (n+1)\left(\frac{n}{2} + 1\right) \\ &= \frac{(n+1)(n+2)}{2}, \end{aligned}$$

as desired.

**2. Recurrence.** Consider the following recurrence relation:

$$\boxed{p_n = p_{n-1} + n.}$$

(a) **Assume** that $p_0 = 3$. In this case, make a table of $p_n$ for $n$ between 0 and 6.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $p_n$ | 3 | 4 | 6 | 9 | 13 | 18 | 24 |

(b) **Assume** that $p_5 = 20$. In this case, tell me the value of $p_2$.

We can rearrange the recurrence to get $p_{n-1} = p_n - n$. Then we have

$$\begin{aligned} p_4 &= p_5 - 5 = 20 - 5 = 15 \\ p_3 &= p_4 - 4 = 15 - 4 = 11 \\ p_2 &= p_3 - 3 = 11 - 3 = 8. \end{aligned}$$

(c) **Assume** that $p_0 = 1$. In this case, tell me a **closed formula** for $p_n$. [Hint: Prob 1.]

We have

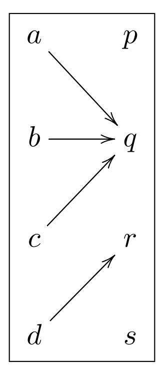
$$\begin{aligned} p_0 &= 1 \\ p_1 &= p_0 + 1 = 1 + 1 \\ p_2 &= p_1 + 2 = 1 + 1 + 2 \\ p_3 &= p_2 + 3 = 1 + 1 + 2 + 3 \\ &\vdots \\ p_n &= 1 + (1 + 2 + 3 + \cdots + n). \end{aligned}$$

Then using the formula from Problem 1 gives $p_n = 1 + \frac{n(n+1)}{2}$.

**3. Functions.** Consider the sets $X = \{a, b, c, d\}$ and $Y = \{p, q, r, s\}$.

(a) Draw an example of a function $f : X \to Y$ that is **not** injective and **not** surjective.

There are 232 different correct answers. Here is one of them:

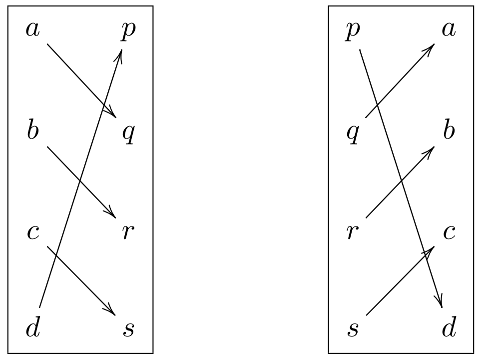



(b) Tell me the total number of different functions from $X$ to $Y$.

In general the number of different functions from $X$ to $Y$ is $\#Y^{\#X}$. In this case we have $\#X = \#Y = 4$, hence there are $4^4 = 256$ functions.

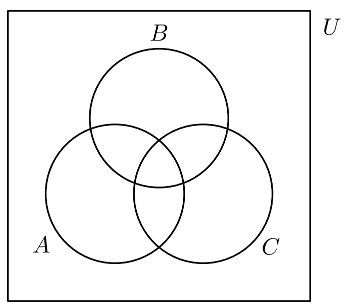
(c) Draw an example of an **invertible** function $f : X \to Y$. Also draw its inverse $f^{-1}$.

There are 24 different correct answers. Here is one of them:

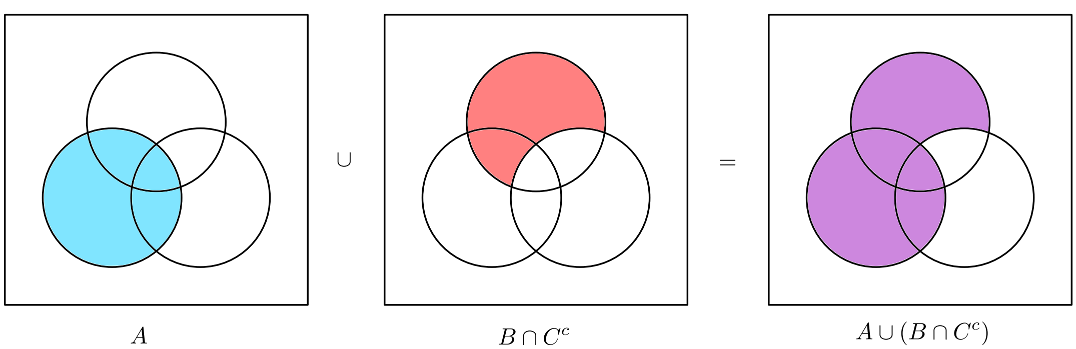



**4. Venn Diagrams.** Fix a universal set $U$ and consider sets $A, B, C \subseteq U$. In this problem you will use Venn diagrams to prove that $A \cup (B \cap C^c) = (A \cup B) \cap (A \cup C^c)$.

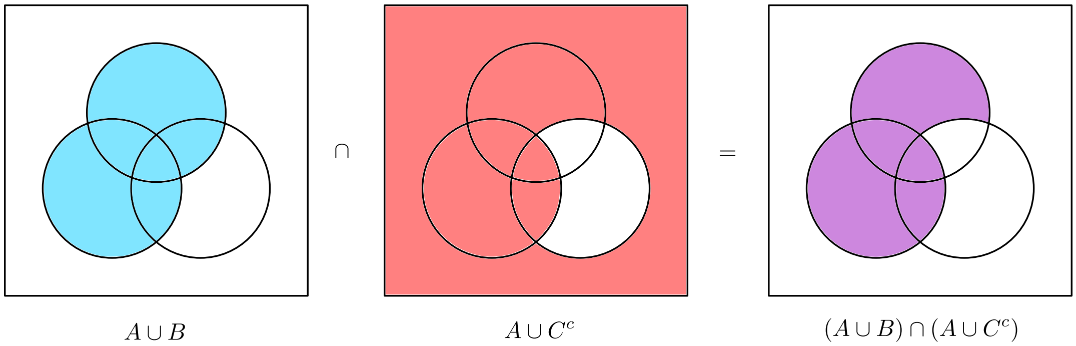
(a) Draw a general Venn diagram displaying the sets $A$, $B$, $C$, and $U$.




(b) Use Venn diagrams to draw the set $A \cup (B \cap C^c)$. Show intermediate steps.




(c) Use Venn diagrams to draw the set $(A \cup B) \cap (A \cup C^c)$. Show intermediate steps.




[Remark: The fact that $A \cup (B \cap C^c) = (A \cup B) \cap (A \cup C^c)$ is an example of the distributive law.]

**5. Truth Tables.** Let $P$ and $Q$ be logical statements.

(a) Draw the truth table for the statement $P \wedge Q$ (i.e., $P$ AND $Q$).

| $P$ | $Q$ | $P \wedge Q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ |

(b) Draw the truth table for the statement $\neg(P \wedge Q)$ (i.e., NOT ($P$ AND $Q$))

| $P$ | $Q$ | $P \wedge Q$ | $\neg(P \wedge Q)$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ |
| $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $T$ |

(c) Draw the truth table for the statement $\neg P \vee \neg Q$ (i.e., (NOT $P$) OR (NOT $Q$)). [Hint: It may help to think about the Venn diagram of the set $A^c \cup B^c$.]

| $P$ | $Q$ | $\neg P$ | $\neg Q$ | $\neg P \vee \neg Q$ |
|---|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ |

[Remark: The fact that $\neg(P \wedge Q) = \neg P \vee \neg Q$ is an example of de Morgan's law.]

**Math 309** **Summer 2014**
**Quiz 1 Solutions** **Drew Armstrong**

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**1.** Let $n$ be a positive whole number. Find a **closed form** for the following sum:

$$1+2+3+\cdots+n = \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n$$

**2.** Let $n$ be a positive whole number. Find a **closed form** for the following sum:

$$1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

**3.** Translate the following statement into English. (Many correct answers.)

$$\forall\, x \in S, P(x)$$

"**Every** element $x$ of the set $S$ satisfies property $P(x)$."

**4.** Translate the following statement into English. (Many correct answers.)

$$\exists\, x \in S, \neg P(x)$$

"There **exists** an element $x$ of the set $S$ such that property $P(x)$ does **not** hold."

**5.** What is the logical relationship between the statements in Problems 3 and 4?

They are opposites. Writing symbolically, we have

$$\neg(\forall\, x \in S, P(x)) \equiv \exists\, x \in S, \neg P(x).$$

**Math 309** **Summer 2014**
**Quiz 2 Solutions** **Drew Armstrong**

Throughout this quiz, $P$ and $Q$ are Boolean variables.

**1.** Write out the truth table for $P \vee Q$.

*Proof.*

| $P$ | $Q$ | $P \vee Q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ |

□

**2.** Write out the truth table for $P \wedge Q$.

*Proof.*

| $P$ | $Q$ | $P \wedge Q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ |

□

Now consider the Boolean function $\varphi : \{T, F\}^2 \to \{T, F\}$ defined by the following table:

| $P$ | $Q$ | $\varphi(P, Q)$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |

**3.** Write out the truth table for $\neg\, \varphi(P, Q)$.

*Proof.*

| $P$ | $Q$ | $\neg\, \varphi(P, Q)$ |
|---|---|---|
| $T$ | $T$ | $F$ |
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ |

□

**4.** Tell me a **formula** for $\neg\, \varphi(P, Q)$ in terms of $\vee, \wedge, \neg$. [Hint: Disjunctive normal form.]

*Proof.* The only $T$ is in the $(T, F)$ row of the truth table. This row corresponds to the formula $P \wedge \neg Q$ (and we can think of it as a region of the corresponding Venn diagram). Therefore the disjunctive normal form is

$$\neg\, \varphi(P, Q) = P \wedge \neg Q.$$

□

**5.** Now tell me a **formula** for $\varphi(P, Q)$. [Hint: Use Problem 4 and de Morgan's identity.]

*Proof.* We can read the disjunctive normal form of $\varphi(P, Q)$ from the truth table above. There are three $T$'s in the rows $(T, T)$, $(F, T)$ and $(F, F)$. Therefore the disjunctive normal form is

$$\varphi(P, Q) = (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q).$$

However, this doesn't look very nice. We get a more compact formula by using Problem 4 and applying de Morgan's identity. We have

$$\begin{aligned} \neg\,\varphi(P, Q) &= P \wedge \neg Q \\ \varphi(P, Q) &= \neg(P \wedge \neg Q) \\ \varphi(P, Q) &= \neg P \vee \neg\neg Q \\ \varphi(P, Q) &= \neg P \vee Q. \end{aligned}$$

□

[Remark: The Boolean function $\varphi$ is very important in mathematics. In this class we will denote it by "$P \Rightarrow Q$" $:= \varphi(P, Q)$ and we will read the statement "$P \Rightarrow Q$" as "if $P$ then $Q$", or "$P$ implies $Q$". (WARNING: This definition might not **totally** agree with your intuition about the words "if ... then ...". For example, do you regard "if $1 + 1 = 3$ then $1 + 1 = 5$" to be a true statement? I do.) Mathematical proofs and logical arguments are built from these arrows.]

**Math 309** **Summer B 2018**
**Quiz 1** **Drew Armstrong**

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**1.** Let $n$ be a positive integer. Tell me a **closed formula** for the following sum:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

**2.** Use your answer from Problem 1 to simplify the following sum:

$$\sum_{k=1}^{n}(2k+1) = 2\left(\sum_{k=1}^{n} k\right) + \left(\sum_{k=1}^{n} 1\right) = 2\cdot\frac{n(n+1)}{2} + n = n(n+1)+n = n(n+2)$$

**3.** Suppose that the numbers $p_n$ are defined by the initial condition $p_0 = 1$ and the recurrence $p_{n+1} = p_n + n + 2$ for all $n \geq 0$. Fill in the following table:

| $n$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $p_n$ | 1 | 3 | 6 | 10 | 15 |

**4.** Continuing from Problem 3, use your answer from Problem 1 to find a **closed formula**:
There are many ways to do this. Here's the least clever way:

$$\begin{aligned}
p_0 &= 1\\
p_1 &= 1 + (0+2)\\
p_2 &= 1 + (0+2) + (1+2)\\
&\vdots\\
p_n &= 1 + (0+2) + (1+2) + (3+2) + (4+2) + \cdots + ((n-1)+2)\\
&= 1 + \sum_{k=0}^{n-1}(k+2)\\
&= 1 + \left(\sum_{k=0}^{n-1} k\right) + \left(\sum_{k=0}^{n-1} 2\right)\\
&= 1 + \frac{(n-1)n}{2} + 2n\\
&= \frac{2 + (n-1)n + 4n}{2}\\
&= \frac{n^2+3n+2}{2}\\
&= \frac{(n+1)(n+2)}{2}
\end{aligned}$$

**Math 309** **Summer B 2018**
**Quiz 2** **Drew Armstrong**

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**0. (Make-up for Quiz 1)** The numbers $p_n$ are defined by the initial condition $p_0 = 1$ and the recurrence $p_n = p_{n-1} - n$ for all $n \geq 1$. Find a closed formula:

$$\begin{aligned} p_n &= 1 - 1 - 2 - 3 - \cdots - n \\ &= 1 - (1 + 2 + 3 + \cdots + n) \\ &= 1 - \frac{n(n+1)}{2} \end{aligned}$$

**1.** Let $S$ be a set and for all elements $x \in S$ let $P(x)$ be a logical statement. Translate the following statements into English:

- "$\forall x \in S, P(x)$"

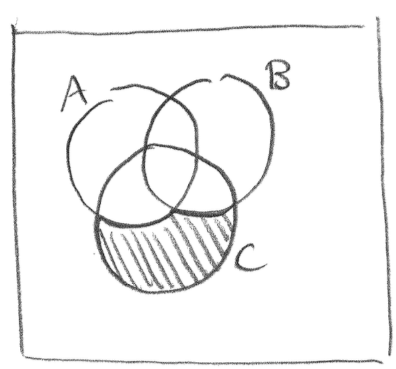
"For all elements $x$ in $S$, the statement $P(x)$ is true."
or "The statement $P(x)$ holds for every element $x$ in $S$."
or "Every element $x$ of $S$ satisfies $P(x)$."

- "$\exists x \in S, \neg P(x)$"

"There exists an element $x$ in $S$ such that $P(x)$ is false."

**2.** Let $A, B, C$ be subsets of the universal set $U$. Use a Venn diagram to illustrate the set

$$(A \cup B)^c \cap C.$$



**3.** Complete the following truth table:

| $P$ | $Q$ | $P \wedge Q$ | $P \vee Q$ | $P \Rightarrow Q$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $F$ | $T$ |

**4.** Consider the Boolean function $\varphi(P, Q)$ defined as follows:

| $P$ | $Q$ | $\varphi(P, Q)$ |
|---|---|---|
| $T$ | $T$ | $F$ |
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $T$ |

Write down two different algebraic formulas for this function:

(There are infinitely many correct answers.)

$$\varphi(P, Q) = (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$$

$$\varphi(P, Q) = \neg Q$$

**Problem 1.**

(a) Draw Pascal's Triangle down to the 7th row.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

(b) Use the triangle to expand $(1+x)^7$.

$$(1+x)^7 = 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$$

(c) Use the triangle to evaluate the following sum:

$$
\begin{aligned}
\sum_{k=0}^{4} (-1)^k \binom{7}{k} &= \binom{7}{0} - \binom{7}{1} + \binom{7}{2} - \binom{7}{3} + \binom{7}{4} \\
&= 1 - 7 + 21 - 35 + 35 \\
&= 15
\end{aligned}
$$

**Problem 2.** Let the sequence $S_0, S_1, S_2, \ldots$ be defined by the following initial condition and recurrence relation:

$$S_n := \begin{cases} 1 & \text{if } n = 0, \\ S_{n-1} + 2^{n-1} & \text{if } n \geq 1. \end{cases}$$

(a) Fill in the following table:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $S_n$ | 1 | 2 | 4 | 8 | 16 | 32 |

(b) Try to guess a simple formula for $S_n$.

I guess that $S_n = 2^n$ for all $n \geq 0$.

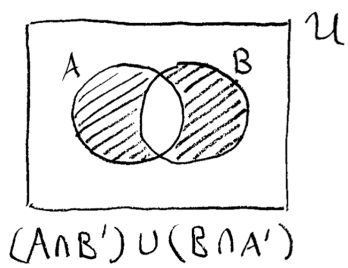
(c) Use induction to prove that your formula is correct.

- *Base Case.* If $n = 0$ then we have $S_0 = 1 = 2^0$. ✓
- *Induction Step.* Now fix some $n \geq 0$ and assume for induction that $S_n = 2^n$. In this case we want to prove that $S_{n+1} = 2^{n+1}$. Indeed, we observe that

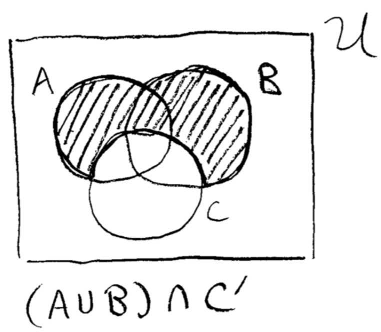
$$\begin{aligned} S_{n+1} &= S_n + 2^n && \text{by definition} \\ &= 2^n + 2^n && \text{by assumption} \\ &= 2 \cdot 2^n \\ &= 2^{n+1}. \end{aligned}$$

**1. Venn Diagrams.** Let $A, B, C \subseteq U$ be subsets of the universal set.

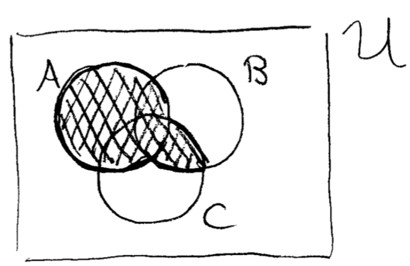
(a) Draw a Venn diagram to show the set $(A \cap B') \cup (B \cap A')$.




(b) Draw a Venn diagram to show the set $(A \cup B) \cap C'$.




(c) Tell the name of the following set (many correct answers):




The shortest name of the set is $A \cup (B \cap C)$. The disjunctive normal form is

$$(A \cap B \cap C) \cup (A \cap B' \cap C) \cup (A \cap B \cap C') \cup (A \cap B' \cap C') \cup (A' \cap B \cap C).$$

Or we can compute the disjunctive normal form of the unshaded region. Then we take the complement and apply de Morgan's law:

$$\begin{aligned} &\left[(A' \cap B' \cap C) \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C')\right]' \\ &= (A' \cap B' \cap C)' \cap (A' \cap B \cap C')' \cap (A' \cap B' \cap C')' \\ &= (A \cup B \cup C') \cap (A \cup B' \cup C) \cap (A \cup B \cup C). \end{aligned}$$

The resulting expression is called the *conjunctive normal form*.

**2. Boolean Functions.** Consider the Cartesian product set:

$$\{T, F\}^n := \underbrace{\{T, F\} \times \{T, F\} \times \cdots \times \{T, F\}}_{n \text{ times}}.$$

(a) Tell me the number of elements of the set $\{T, F\}^n$.

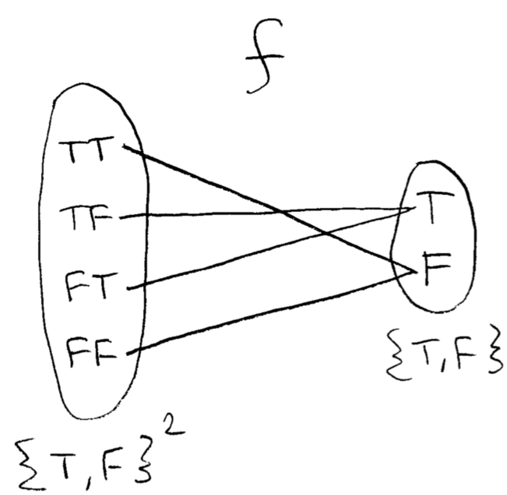
$$\#\{T, F\}^n = \underbrace{\#\{T, F\} \times \#\{T, F\} \times \cdots \times \#\{T, F\}}_{n \text{ times}} = \underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}} = 2^n$$

(b) Count the functions $f : \{T, F\}^2 \to \{T, F\}$ with two inputs and one output.

The number of functions from $A$ and $B$ is $(\#B)^{\#A}$. Hence the number of functions from $\{T, F\}^2$ to $\{T, F\}$ is

$$(\#\{T, F\})^{(\#\{T,F\}^2)} = 2^{(2^2)} = 2^4 = 16.$$

(c) Fill in the arrows for the function $f(P, Q) = (P \wedge \neg Q) \vee (Q \wedge \neg P)$:




Remark: This is just one of the 16 Boolean functions with two inputs and one output. It is commonly called XOR. This function also appeared in Problem 1(a).