8/25/14

Welcome to MTH 309

Course Info:

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- office hours: TBA.

There is no required textbook. I will scan all lecture notes and post them on my webpage:

www.math.miami.edu/~armstrong

There will be ≈ 6 HW assignments
3 in-class exams
NO FINAL EXAM.

Your grade will be based on

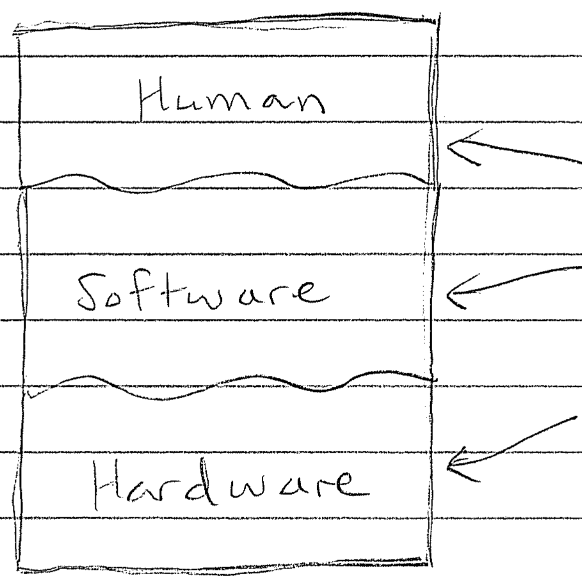
25%	Homework
25%	Exam 1
25%	Exam 2
25%	Exam 3
100%	

Course Topic: Discrete Mathematics

"Discrete", as opposed to "Continuous"
(Calculus)

This subject is particularly relevant for computer applications because computers are essentially discrete structures. Any continuous problem must first be "discretized" before it can be implemented on a computer.

Computers have various levels:



Different levels require different kinds of mathematics

1s & 0s

Some major themes:

- 0s & 1s
- Logic & "Boolean Algebra"
- Counting & Probability
- Induction & Recursion
- How is reasoning performed by humans & computers?

I will try to weave this into some kind of story, to make it more fun.

BEGIN.

Let's start with a mathematical problem.

Problem (Jacob Steiner, 1826):

What is the maximum number of regions formed by n (infinite) lines in the (infinite) plane?

Let L_n denote this maximum number of regions.

Our Goal: <u>solve</u> for L_n

What does this mean?

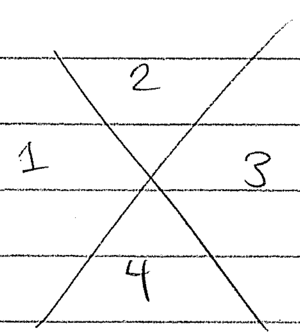
- give an algorithm to compute L_n
- give an "efficient" algorithm to compute L_n
- give an algebraic "formula" for L_n.
- give a "nice" algebraic formula for L_n.
- say how fast L_n grows with n.

⋮

we'll see

To analyze a discrete problem we <u>always</u> start with <u>experiments</u>

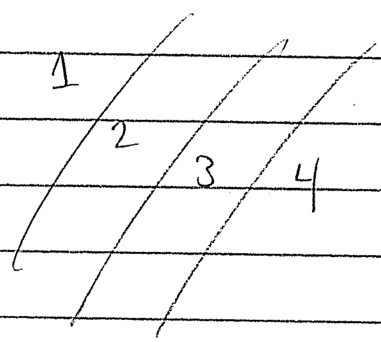
Obviously $L_0 = 1$.
$L_1 = 2$

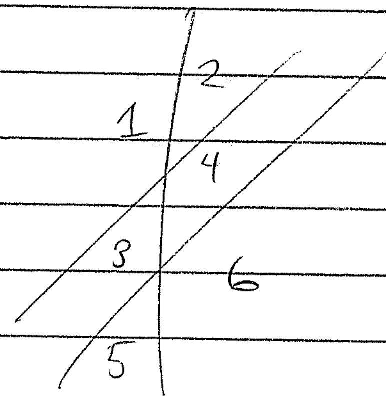


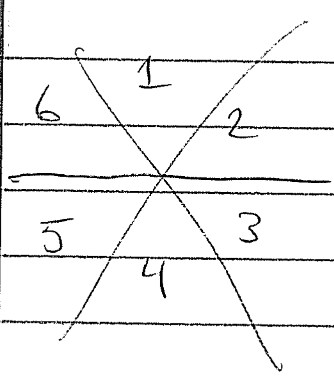
$L_2 = 4$

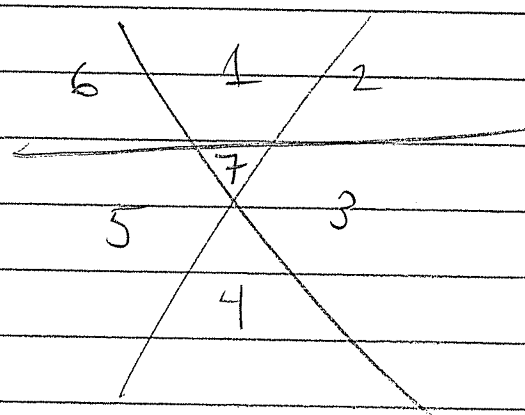
[Guess ("conjecture"): $L_n = 2^n$?]

Continue ... $L_3 = ?$









It looks like $L_3 = 7$

[Note: $7 \neq 2^3$, so our guess was wrong. Oh well. Do we have a new guess?]

Now you experiment. Try to find L_4.

I hope you found $L_4 = 11$.

Here's our data

n	0	1	2	3	4	5
L_n	1	2	4	7	11	?

Can we predict L_5 without drawing any pictures?

We might notice that

$$L_1 = L_0 + 1$$
$$L_2 = L_1 + 2$$
$$L_3 = L_2 + 3$$
$$L_4 = L_3 + 4$$

and then we might guess that

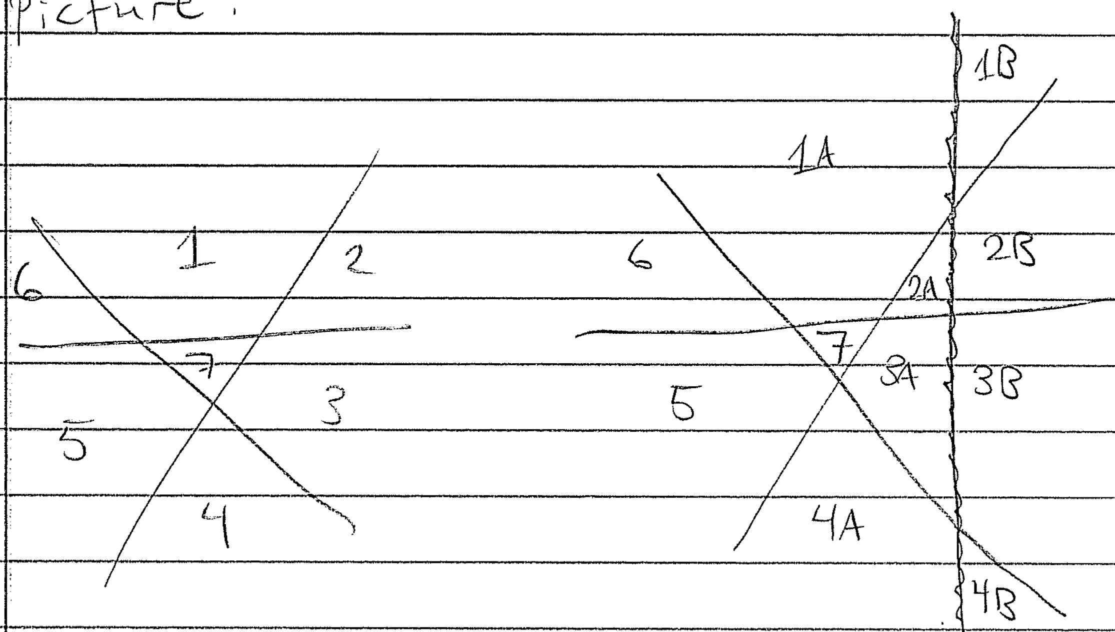
$$\boxed{L_n = L_{n-1} + n \quad \text{for all } n \geq 1}$$

This is called a "recurrence relation".

<u>Why</u> might this relation be true?

Suppose we already have $n-1$ lines.
When we add the n^{th} line we will get
1 <u>new</u> region for each old region that
the new line cuts through.

Picture:



If the new line cuts through n
old regions, I guess we will get

$$L_n = L_{n-1} + n .$$

old regions formed by $n-1$ lines

new regions created by the n^{th} line.

That's not really a proof but at least it's plausible.

For now let's just assume that

- $L_0 = 1$
- $L_n = L_{n-1} + n$ for $n \geq 1$.

This is a recurrence relation with an initial condition (analogous to a differential equation).

Can we solve the recurrence to find a "formula" for L_n?

$$L_1 = L_0 + 1 = 1 + 1$$
$$L_2 = L_1 + 2 = 1 + 1 + 2$$
$$L_3 = L_2 + 3 = 1 + 1 + 2 + 3.$$
$$\vdots$$
$$L_n = 1 + 1 + 2 + 3 + \cdots + n$$
$$= 1 + (1 + 2 + 3 + \cdots + n).$$

$$\boxed{L_n = 1 + \sum_{k=1}^{n} k}$$

Yay. We found a formula. But is it a "nice" formula?

Not really. The $\sum_{k=1}^{n} k$ part is not nice.

But maybe we can't do any better? This problem is nice because there's a <u>lucky</u> <u>trick</u>

Let $S_n := \sum_{k=1}^{n} k = 1+2+3+\cdots+n$.

Theorem (Carl Friedrich Gauss, when he was 6 years old.):

$$S_n = \frac{n(n+1)}{2}.$$

<u>Proof</u>: Consider the quantity $2S_n$.

$$\begin{aligned} 2S_n &= S_n + S_n \\ &= (1 + 2 + 3 + \cdots + n) \\ &\quad + (n + n-1 + n-2 + \cdots + 1) \\ &= \underbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1)}_{n \text{ times}}. \end{aligned}$$

So $2S_n = n(n+1)$, hence

$$S_n = \frac{n(n+1)}{2}.$$

Finally, this gives us a "closed formula" for L_n.

$$\begin{aligned} L_n &= 1 + S_n \\ &= 1 + \frac{n(n+1)}{2} \\ &= \frac{2 + n(n+1)}{2} \\ &= \frac{n^2+n+2}{2} \end{aligned}$$

Conclusion:

$$\boxed{L_n = \frac{n^2+n+2}{2}}$$

Is that a "nice" formula? Yes!

Bonus: This also gives us an "asymptotic exstimate" for L_n.

For large n we have

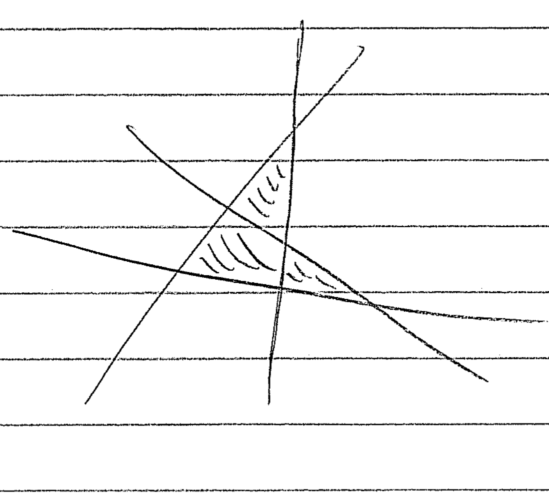
$$L_n \sim \frac{1}{2} n^2 .$$

[This just means that

$$\lim_{n \to \infty} \frac{L_n}{n^2/2} = 1 .]$$

Thinking Problem:

What is the maximum number of bounded regions created by n lines in the plane?



3 bounded regions created by 4 lines.

8/27/14

No HW yet.
Office Hours: still TBA

Last time we considered a problem from Jacob Steiner (1826).

Let $L_n :=$ The maximum number of regions that can be formed by n lines in the plane.

We found that L_n satisfies the following recurrence and initial condition

$$\boxed{\begin{array}{l} \bullet\ L_0 = 1 \\ \bullet\ L_n = L_{n-1} + n \quad \text{for } n \geq 1 \end{array}}$$

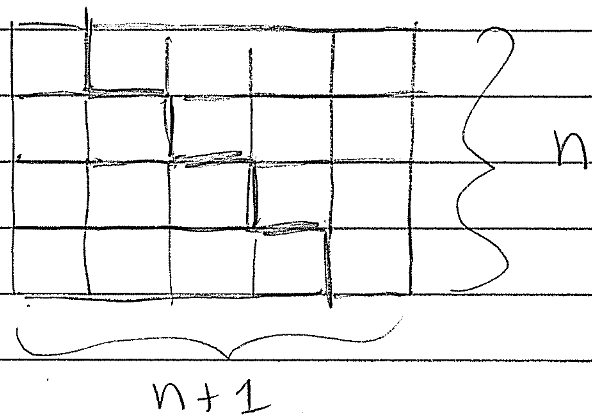
We expanded this to obtain

$$L_n = 1 + (1+2+3+\cdots+n)$$
$$= 1 + \sum_{k=1}^{n} k$$

and then we used a trick of Carl Friedrich Gauss to show that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Here's the trick: Note that $\sum_{k=1}^{n} k$ is the area of a "staircase" made of 1×1 squares



Two such staircases fit together to form an $n \times (n+1)$ rectangle. Hence

2(area of staircase) = area of rectangle

$$2\sum_{k=1}^{n} k = n(n+1)$$

$$\sum_{k=1}^{n} k = n(n+1)/2$$

//

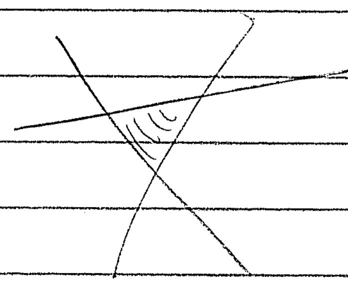
We conclude that the max # regions formed by n lines in the plane is

$$L_n = 1 + \frac{n(n+1)}{2}.$$

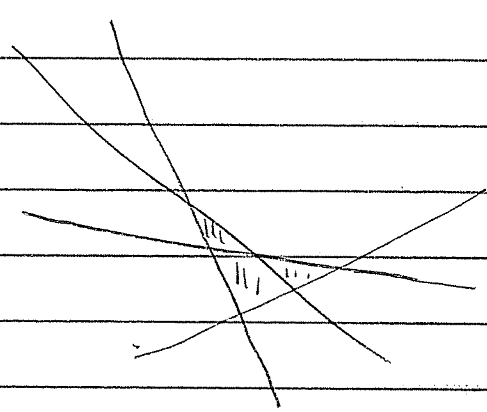
Now here's a related problem:

Find the max. # of bounded regions formed by n lines in the plane. Let B_n be this number.

$B_0 = 0$
$B_1 = 0$
$B_2 = 0$



$B_3 = 1$



$B_4 = 3.$

Any guesses for B_n ?

(Insert your guess here.)

Can we "prove" that our guess is correct? I think this is tricky: I suggest that we look at unbounded regions instead.

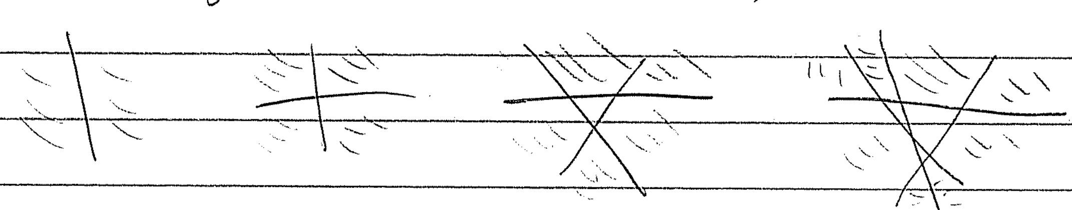
Do we have a guess for counting unbounded regions?

(Insert guess here.)

Can we prove it? Let's see.

Consider n lines in the plane such that no two are parallel. I claim that these lines create exactly $2n$ unbounded regions.

(Do an experiment until you believe it.)



2 4 6 8 etc.

OK, it's true for small n. What about large n, say $n = 1000000000000$?

We'll use a powerful method called "induction". (This method will reappear later and play a big role in the course.)

We'll show that if the statement is true for n lines, then the statement is true for $n+1$ lines.

So assume that any n lines in the plane (no two parallel) make $2n$ unbounded regions. (Did you assume it yet? I'll wait.)

Good. Now we want to show that any $n+1$ lines (no two parallel) make $2(n+1)$ unbounded regions.

How can we show this? Consider any $n+1$ lines

$$\ell_1, \ell_2, \ell_3, \ldots, \ell_{n+1}$$

such that no two are parallel.

[These lines are fixed, but arbitrary. This is a subtle idea and it takes practice to accept it.]

How many unbd. regions do they make? I don't know.

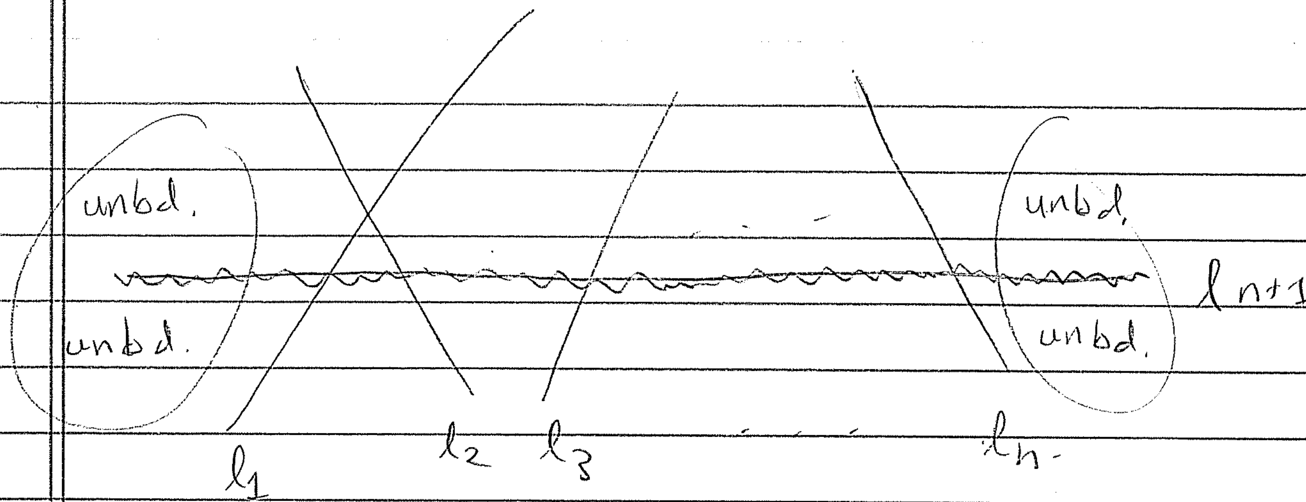
OK, so remove the line l_{n+1}. The lines

$$l_1, l_2, \ldots, l_n$$

still have no two parallel, so they make $2n$ unbounded regions. (Why? Because we assumed it. Did you forget already?)

Great. Now put the line l_{n+1} back where you found it. How many new unbd. regions does it create?

Hmm. We know that l_{n+1} crosses every other line (because no two are parallel). It might look like this:



Note that two new unbounded regions are created on the ends and no new unbd. can be created in the middle. (Don't worry too much about this. Just do a few examples until you find it plausible.)

Therefore, the total # of unbounded regions formed by $l_1, l_2, \ldots, l_{n+1}$ is

$$2n+2 = 2(n+1)$$

as desired. ///

Did you like that? Don't worry; by the end of the semester it will seem normal.

We saw that

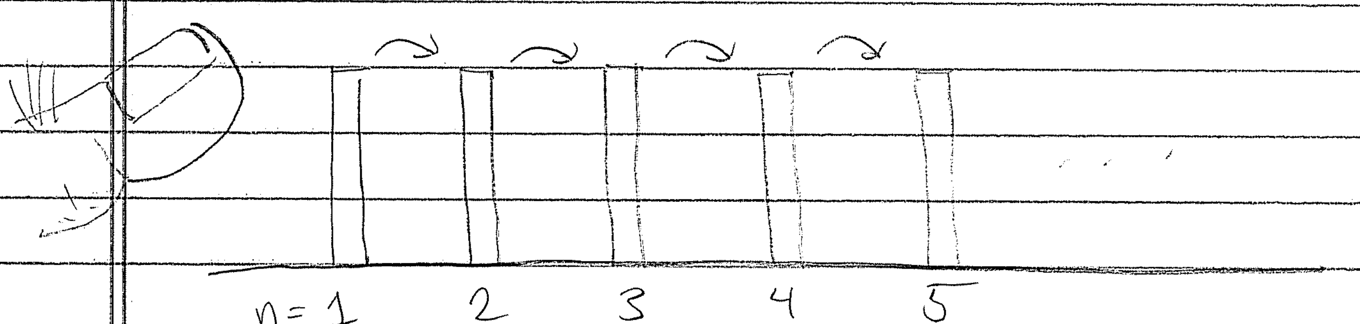
(1) The statement starts out true, and

(2) If it's true at some point then it stays true after that.

That's good enough for me!

We conclude that any n lines (no two parallel) make 2n bounded regions, for all $n \geq 1$

We can visualize what we did in terms of falling "dominoes"



(1) is your finger
(2) is gravity
BOTH STEPS ARE NECESSARY.

Back to bounded regions B_n :

Suppose we have n lines forming the maximum # of regions, i.e.

$$1 + \frac{n(n+1)}{2}$$

Certainly there can be no parallel lines [why not?], so we know that the # of unbounded regions is $2n$.

It follows that the # of bounded regions is

$$1 + \frac{n(n+1)}{2} - 2n$$

$$= (2 + n(n+1) - 4n)/2$$

$$= (2 + n^2 + n - 4n)/2$$

$$= (n^2 - 3n + 2)/2$$

$$= (n-1)(n-2)/2.$$

Is this the max # of bounded regions we can possibly get?

The answer is yes, but I'll let you think about it.

We conclude that

$$B_n = \frac{(n-1)(n-2)}{2}$$

for all $n \geq 1$.

[Remark: This formula fails when $n=0$, but I'm not worried about that.]

Thinking for Next Time:

Can you find a closed formula for the sum of the first n "squares",

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = ?$$

9/3/14

HW 1 due Wed Sept 10 in class,
My office Hours are .
- Mondays 10 – 11 am
- Fridays 4 – 5 pm
- also by appointment

Let $B_n :=$ the max. # of bounded regions that can be formed by n lines in the plane.

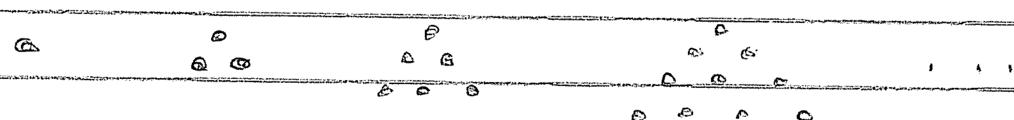
Last time we showed that

$$B_n = \frac{(n-1)(n-2)}{2} \quad \text{for } n \geq 1$$

Table :

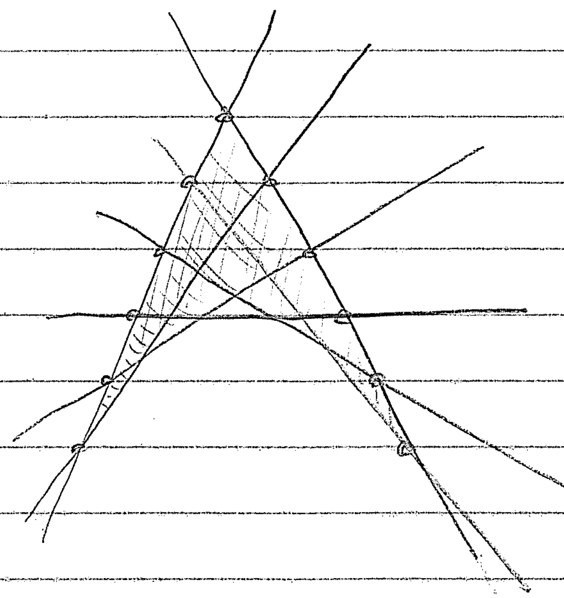
n	1	2	3	4	5	6	7	...
B_n	0	0	1	3	6	10	15	

These are called "triangular numbers"



1 3 6 10

Actually it's not surprising that we got triangular numbers:



(see the "triangle" of bounded regions?)

Before moving to a discussion of logic, I want to bring this story to a natural conclusion.

Steiner (1826) actually considered a more general problem.

Let $P_n :=$ the max # of 3D regions that can be formed by n planes in space.

Goal: "Solve" for P_n.

Step 1: Find a Recurrence.

Suppose we have $n+1$ planes

$$\Pi_1, \Pi_2, \ldots, \Pi_{n+1}$$

and suppose the form the max. # P_n of 3D regions.

If we remove p_{n+1} then the planes

$$\Pi_1, \Pi_2, \ldots, \Pi_n$$

will form the max. # of 3D regions P_{n-1}.

Q: How many new 3D regions are formed when we put the plane Π_{n+1} back?

A: One new region for each old region that Π_{n+1} cuts through. How many old regions does Π_{n+1} cut through?

Note that the planes $\Pi_1, \Pi_2, \ldots, \Pi_n$ intersect the plane Π_{n+1} in n lines. These n lines form some 2D regions in Π_{n+1} and these 2D regions correspond exactly to the 3D regions that Π_{n+1} cuts through.

Q: How many 2D regions are formed by the n lines in Π_{n+1}?

A: The maximum possible.

Q: How many is that?

A: We proved that the answer is

$$L_n = 1 + \frac{n(n+1)}{2}$$

$$= \frac{1}{2}n^2 + \frac{1}{2}n + 1 .$$

We conclude that

$$P_{n+1} = P_n + L_n$$

$$= P_n + \frac{1}{2}n^2 + \frac{1}{2}n + 1 .$$

Thus we have a recurrence relation and initial condition:

$$\boxed{\begin{array}{l} \bullet\ P_0 = 1 \\ \bullet\ P_{n+1} = P_n + L_n \quad \text{for } n \geq 0 \end{array}}$$

Now, can we "solve" for P_n?

Let's Try:

$$P_0 = 1$$
$$P_1 = P_0 + L_0 = 1 + L_0.$$
$$P_2 = P_1 + L_1 = 1 + L_0 + L_1$$
$$P_3 = P_2 + L_2 = 1 + L_0 + L_1 + L_2$$
$$\vdots$$
$$P_n = 1 + L_0 + L_1 + L_2 + \cdots + L_{n-1}$$

$$P_n = 1 + \sum_{k=0}^{n-1} L_k$$

But we already have a closed formula for L_k,

$$L_k = \frac{1}{2}k^2 + \frac{1}{2}k + 1.$$

Let's plug it in.

$$P_n = 1 + \sum_{k=0}^{n-1} \left(\frac{1}{2}k^2 + \frac{1}{2}k + 1 \right)$$

I can rearrange this sum, right? Yes.

$$P_n = 1 + \frac{1}{2}\sum_{k=0}^{n-1} k^2 + \frac{1}{2}\sum_{k=0}^{n-1} k + \sum_{k=0}^{n-1} 1 .$$

Can we simplify further?

- $$\sum_{k=0}^{n-1} 1 = \underbrace{1 + 1 + 1 + \cdots + 1}_{n \text{ times}} = n.$$

- $$\sum_{k=0}^{n-1} k = \cancel{0} + 1 + 2 + \cdots + (n-1)$$
  $$= \sum_{k=1}^{n-1} k = \frac{(n-1)(n-1+1)}{2} = \frac{n(n-1)}{2}$$
  Gauss

- $$\sum_{k=0}^{n-1} k^2 = \cancel{0^2} + 1^2 + 2^2 + \cdots + (n-1)^2$$
  $$= \sum_{k=1}^{n-1} k^2 = ?$$

But now I'm stuck.

Let's see what we have so far.

$$P_n = 1 + \frac{1}{2}\left[\sum_{k=1}^{n-1} k^2\right] + \frac{1}{2}\,\frac{n(n-1)}{2} + n$$

$$= \frac{1}{2}\left[\sum_{k=1}^{n-1} k^2\right] + \frac{4 + n(n-1) + 4n}{4}$$

$$= \frac{1}{2}\left(\sum_{k=0}^{n-1} k^2\right) + \frac{n^2 + 3n + 4}{4}$$

Now we're stuck. ☹

What can we do?

Name the problem. (This strategy is called "name and conquer".)

For all positive integers p and n we define the sum of the first n pth powers.

$$S_p(n) := 1^p + 2^p + 3^p + \cdots + n^p$$

$$= \sum_{k=1}^{n} k^p.$$

We wont $S_2(n-1)$, but hey it might be nice to have a formula for general p and n (mightn't it?)

I'll make you a deal: I'll just tell you the answer for $S_2(n)$, but then you have to prove it's correct.

Deal? OK.

Claim: for all $n \geq 1$ we have

$$S_2(n) = \frac{n(n+1)(2n+1)}{6}.$$

Proof: When there is no other option we use the method of induction.

(1) Check it works for small cases!

$$1^2 = S_2(1) = \frac{1(1+1)(2\cdot 1+1)}{6} = \frac{1\cdot 2\cdot 3}{6} = 1. \checkmark$$

$$1^2+2^2 = S_2(2) = \frac{2(2+1)(2\cdot 2+1)}{6} = \frac{2\cdot 3\cdot 5}{6} = 5 \checkmark$$

$$1^2+2^2+3^2 = S_2(3) = \frac{3(3+1)(2\cdot 3+1)}{6} = \frac{3\cdot \overset{2}{\cancel{4}}\cdot 7}{\cancel{6}} = 14 \checkmark$$

Eventually we (or our computer, or whatever) will get tired of this. It is not physically possible to check infinitely many cases.

So we proceed to step 2.

(2) The induction trick.

We'll show that **if** $S_2(n) = \frac{n(n+1)(2n+1)}{6}$ is true, **then**

$$S_2(n+1) = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$$
$$= \frac{(n+1)(n+2)(2n+3)}{6}$$

is also true. We begin by assuming

$$1^2 + 2^2 + \cdots + n^2 = S_2(n) = \frac{n(n+1)(2n+1)}{6}.$$

Did you assume it yet? (I'll wait.)

[Question: What is n here? Is it a constant, a variable, or something in between?]

In this hypothetical case, we will investigate $S_2(n+1)$. Note that

$$S_2(n+1) = 1^2 + 2^2 + 3^2 + \cdots + (n+1)^2$$

$$= \left(1^2 + 2^2 + \cdots + n^2\right) + (n+1)^2$$

$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= (n+1)\left[\frac{n(2n+1)}{6} + n+1\right]$$

$$= (n+1)\left[\frac{2n^2+n}{6} + \frac{6n+6}{6}\right]$$

$$= \frac{(n+1)(2n^2+7n+6)}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6}$$

as desired. Yay, we're done.

We have shown that

① The formula starts out true.
② If the formula is true at some point then it stays true after that.

Finally, we can finish Steiner's problem.

$$P_n = 1 + \frac{1}{2}S_2(n-1) + \frac{1}{2}\frac{n(n-1)}{2} + n.$$

$$= (n+1) + \frac{1}{2}\frac{(n-1)n(2n-1)}{6} + \frac{1}{2}\frac{n(n-1)}{2}$$

$$= (n+1) + \frac{n(n-1)}{2}\left[\frac{2n-1}{6} + \frac{1}{2}\right]$$

$$= (n+1) + \frac{n(n-1)}{2}\left[\frac{2n-1+3}{6}\right]$$

$$= (n+1) + \frac{n(n-1)}{2}\left(\frac{2n+2}{6}\right)$$

$$= (n+1) + \frac{n(n-1)(n+1)}{6}$$

$$= (n+1)\left[1 + \frac{n(n-1)}{6}\right]$$

That's pretty nice right?

Thinking Problems:

1. What is the nicest way to write the formula for P_n?

2. What is the max. # of <u>bounded</u> 3D regions formed by n planes?

3. How could you guess the formula

$$S_2(n) = \frac{n(n+1)(2n+1)}{6} \quad ?$$

9/8/14

HW 1 due Wed (beginning of class)
HW 2 will be due Mon Sept 22.
Exam 1 will be on Wed Sept 24

Soon we will shift to a discussion of LOGIC.
But first I want to finish the stories of ① Steiner's Problem and ② Sums of p^{th} powers.

① Recall that

$P_n :=$ max. # 3D regions formed by n planes in space

$L_n :=$ max. # 2D regions formed by n lines in the plane

Last time we argued that

$$P_n = P_{n-1} + L_{n-1}.$$

Together with initial condition $P_0 = 1$, this gives

$P_0 = 1$

$P_1 = P_0 + L_0 = 1 + L_0$

$P_2 = P_1 + L_1 = 1 + L_0 + L_1$

$P_3 = P_2 + L_2 = 1 + L_0 + L_1 + L_2$

$\vdots$

$P_n = 1 + L_0 + L_1 + L_2 + \cdots + L_{n-1}$.

$$P_n = 1 + \sum_{k=0}^{n-1} L_k$$

Since we already know the formula

$$L_k = 1 + \frac{k(k+1)}{2} = \frac{1}{2}k^2 + \frac{1}{2}k + 1$$

we can plug this in to get

$$P_n = 1 + \sum_{k=0}^{n-1}\left(\frac{1}{2}k^2 + \frac{1}{2}k + 1\right)$$

$$= 1 + \frac{1}{2}\left(\sum_{k=0}^{n-1} k^2\right) + \frac{1}{2}\left(\sum_{k=0}^{n-1} k\right) + \left(\sum_{k=0}^{n-1} 1\right).$$

Now we have three sums to evaluate. Luckily we know them.

- $$\sum_{k=0}^{n-1} 1 = \underbrace{1 + 1 + 1 + \cdots + 1}_{n \text{ times}} = n.$$

- $\sum_{k=0}^{n-1} k = 0+1+2+\cdots+(n-1) = \frac{(n-1)((n-1)+1)}{2}$

- $\sum_{k=0}^{n-1} k^2 = 0^2+1^2+2^2+\cdots+(n-1)^2 = ?$

$$= \frac{(n-1)((n-1)+1)(2(n-1)+1)}{6}$$

$$= (n-1)(n)(2n-1)/6$$

(we proved this last time.)

Plugging in gives

$$P_n = 1 + \frac{1}{2}\frac{(n-1)(n)(2n-1)}{6} + \frac{1}{2}\frac{(n-1)n}{2} + n.$$

Should we simplify?

$$P_n = (n+1) + \frac{1}{2}n(n-1)\left[\frac{2n-1}{6} + \frac{1}{2}\right]$$

$$= (n+1) + \frac{1}{2}n(n-1)\left[\frac{2n-1+3}{6}\right]$$

$$= (n+1) + \frac{1}{\cancel{2}}n(n-1)\left[\frac{\cancel{2}(n+1)}{6}\right]$$

$$= (n+1)\left[1 + \frac{n(n-1)}{6}\right]$$

$$= (n+1)\left[\frac{n^2 - n + 6}{6}\right]$$

Seems like that's the best we can do. But I'll tell you a secret: I prefer to write it like this.

$$P_n = \frac{n(n-1)(n-2)}{3} + \frac{n(n-1)}{2} + n + 1$$

$$= \binom{n}{3} + \binom{n}{2} + \binom{n}{1} + \binom{n}{0}.$$

Why do I like to write it like this? I'll tell you another secret. The max # of <u>bounded</u> 3D regions formed by n planes in space equals

$$\binom{n}{3} - \binom{n}{2} + \binom{n}{1} - \binom{n}{0}.$$

Proof omitted

Epilogue: Ludwig Schläfli returned to the problem in the 1840s. He showed that the max. # d-dimensional regions formed by n "hyperplanes" in d-dim. space equals

$$\binom{n}{d} + \binom{n}{d-1} + \cdots + \binom{n}{2} + \binom{n}{1} + \binom{n}{0}$$

and the number of <u>bounded</u> regions is

$$\binom{n}{d} - \binom{n}{d-1} + \cdots + (-1)^{d-1}\binom{n}{1} + (-1)^{d}\binom{n}{0}$$

Proof omitted, obviously!

We will return to the numbers

$$\binom{n}{k} := \frac{n!}{k!\,(n-k)!}$$

in a little while.

② Sums of p^{th} powers.

Recall that we defined

$$S_p(n) := 1^p + 2^p + 3^p + \cdots + n^p$$

What do we know so far? We know

- $S_0(n) = n$
- $S_1(n) = \frac{n(n+1)}{2}$ (Gauss' trick)
- $S_2(n) = \frac{n(n+1)(2n+1)}{6}$.

We proved the last one by induction. But where does the formula come from? How would we ever find this ourselves?

This is not so bad actually. The trick is to evaluate the sum

$$S_3(n+1)$$

in two different ways.

First Way:

$$S_3(n+1) = 1^3 + 2^3 + \cdots + (n+1)^3$$

$$= \left(1^3 + 2^3 + \cdots + n^3\right) + (n+1)^3$$

$$= S_3(n) + (n+1)^3$$

Second Way:

$$S_3(n+1) = \sum_{k=1}^{n+1} k^3 = \sum_{k=0}^{n} (k+1)^3$$

$$= \sum_{k=0}^{n} \left(k^3 + 3k^2 + 3k + 1\right)$$

$$= \sum_{k=0}^{n} k^3 + 3\sum_{k=0}^{n} k^2 + 3\sum_{k=0}^{n} k + \sum_{k=0}^{n} 1$$

$$= \sum_{k=1}^{n} k^3 + 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=0}^{n} 1$$

$$= S_3(n) + 3S_2(n) + 3S_1(n) + (n+1).$$

Equate the two expressions for $S_3(n+1)$:

$$\cancel{S_3(n)} + (n+1)^3 = \cancel{S_3(n)} + 3S_2(n) + 3S_1(n) + (n+1)$$

Note that $S_3(n)$ <u>cancels</u>. That's very good because we don't know it (yet).
[That's the <u>trick</u>!]

Now we can <u>solve</u> for $S_2(n)$.

$$3 \cdot S_2(n) = (n+1)^3 - 3 \cdot S_1(n) - (n+1).$$

$$= (n+1)^3 - 3 \cdot \frac{n(n+1)}{2} - (n+1).$$

$$= (n+1)\left[(n+1)^2 - \frac{3n}{2} - 1\right]$$

$$= (n+1)\left[\frac{2(n+1)^2 - 3n - 2}{2}\right]$$

$$= \frac{1}{2}(n+1)\left[2(n^2+2n+1) - 3n - 2\right]$$

$$= \frac{1}{2}(n+1)\left[2n^2 + n\right]$$

$$= \frac{1}{2}(n+1)\,n\,(2n+1).$$

We conclude that

$$3 \cdot S_2(n) = \frac{1}{2} n(n+1)(2n+1)$$

$$S_2(n) = \frac{1}{6} n(n+1)(2n+1).$$

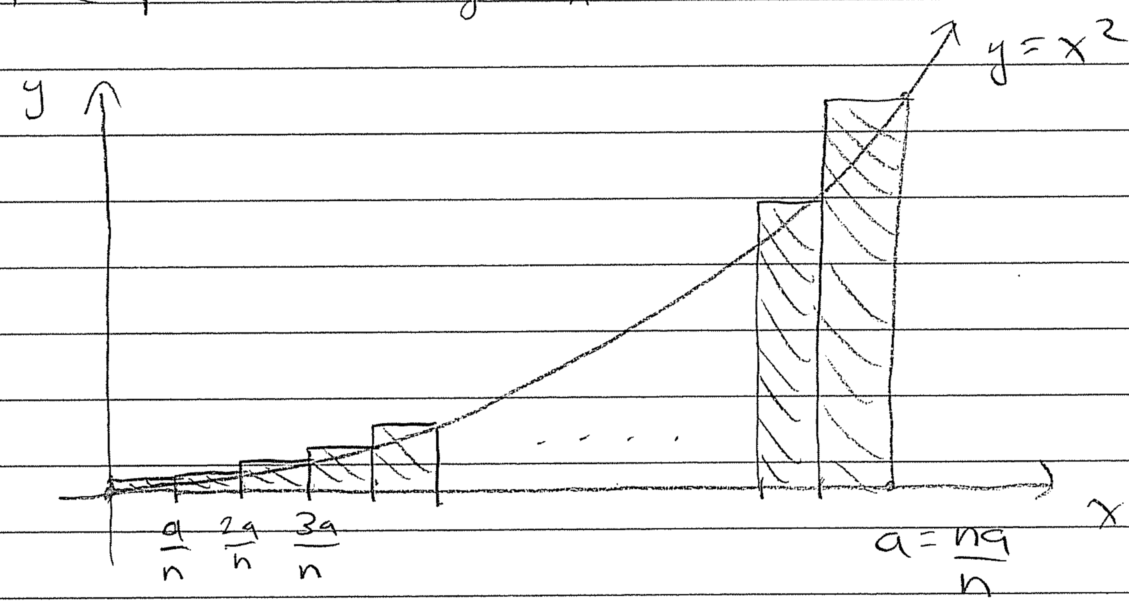
Or we could write it like this:

$$\boxed{S_2(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n}$$

Q: Why would anyone care?

A: Here's why Piere de Fermat cared (1636).

He wanted to compute the area under the parabola $y = x^2$:



Today we call the <u>exact</u> area

$$\int_0^a x^2 \, dx .$$

By appoximating the area with n rectangles,

$$\int_0^a x^2 \, dx \approx \text{sum of areas of rectangles}$$

$$= \sum_{k=1}^{n} \underbrace{\left(\frac{a}{n}\right)}_{\substack{\text{base of} \\ k^{\text{th}} \text{ rectangle}}} \underbrace{\left(\frac{ka}{n}\right)^2}_{\substack{\text{height of} \\ k^{\text{th}} \text{ rectangle}}}$$

$$= \sum_{k=1}^{n} \frac{a}{n} \frac{k^2 a^2}{n^2}$$

$$= \frac{a^3}{n^3} \sum_{k=1}^{n} k^2$$

$$= \frac{a^3}{n^3} \left(\frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n \right)$$

$$= a^3 \left(\frac{1}{3} + \underbrace{\boxed{\frac{1}{2n} + \frac{1}{6n^2}}}_{\text{goes to } 0 \text{ as } n \to \infty} \right)$$

Taking the limit as $n \to \infty$ gives

$$\int_0^a x^2 dx = \lim_{n\to\infty} a^3 \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \right)$$

$$= a^3 \left(\frac{1}{3} + 0 + 0 \right)$$

$$= \frac{a^3}{3}$$

Of course the Fundamental Theorem of Calulus would be faster, but it was not known in 1636. (How do think Newton and Leibniz discovered the FTA anway?)

Remarks :

- More generally, Fermat (1636) showed that $S_p(n) \sim \frac{1}{p+1} n^{p+1}$, and hence

$$\int_0^a x^p dx = \frac{a^{p+1}}{p+1}.$$

- The exact formula for $S_p(n)$ is a little tricky. It involves some funny numbers called "Bernoulli numbers".

They start out like this:

$B_0 = 1, \ B_1 = \frac{1}{2}, \ B_2 = \frac{1}{6}, \ B_3 = 0, \ B_4 = -\frac{1}{30}, \ B_5 = 0,$

$B_6 = \frac{1}{42}, \ B_7 = 0, \ B_8 = -\frac{1}{30}, \ \ldots$

There is a pattern, but it's not very easy to write down. For the curious, here it is:

$$B_p = 1 - \sum_{k=0}^{p-1} \frac{1}{p-k+1} \binom{p}{k} B_k$$

The formula for $S_p(n)$ is called "Faulhaber's formula" and it goes like this:

$$S_p(n) = \frac{1}{p+1} \sum_{k=0}^{p} \binom{p+1}{k} B_k \, n^{p+1-k}$$

[I promise you <u>won't</u> need to know this!]