Math 309 **Fall 2014**
Homework 6 **Drew Armstrong**

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1. Expected Value. Let S be a **finite** sample space with probability function $P : \wp(S) \to \mathbb{R}$ and let $X : S \to \mathbb{R}$ be any random variable.

(a) For all real numbers $k \in \mathbb{R}$ we define the event $E_k = \{s \in S : X(s) = k\}$. This is the set of outcomes that take value k under X. Then we define $P(X = k) := P(E_k)$ and we call this the "probability that X equals k". Explain why

$$P(X = k) = \sum_{s \in E_k} P(s).$$

(The sum is over elements s of the set E_k and we use the notation $P(s) := P(\{s\})$.)

(b) Following Archimedes' "Law of the Lever", we define the **expected value** of the random variable X by

$$E(X) := \sum_k k \cdot P(X = k).$$

Since S is finite, there are only finitely many values k such that $P(X = k) \neq 0$, so we can interpret this as a **finite sum** (and hence avoid any complications with integrals or convergence). Use part (a) to explain why

$$E(X) = \sum_{s \in S} X(s) \cdot P(s),$$

where the sum is over all elements s of the sample space S.

2. Linearity of Expectation. Let X and Y be two random variables on a finite sample space S, and let a and b be constants. We define the random variable $aX + bY$ by $(aX + bY)(s) := aX(s) + bY(s)$ for all $s \in S$.

(a) Use the result of Problem 1 to prove that $E(aX + bY) = aE(X) + bE(Y)$.

(b) Use the result of part (a) to show that

$$E((X - E(X))^2) = E(X(X - 1)) + E(X) - E(X)^2.$$

3. An Urn Problem. An urn contains 6 red balls and 3 green balls. You reach in and grab 4 balls at random. (Assume that each outcome is equally likely.) Let X be the number of red balls that you get.

(a) Compute the probability $P(X = k)$ for each possible value of k.

(b) Compute the expected number of red balls, $E(X)$.

(c) Compute the variance $\text{Var}(X)$. You can use the formula $E((X - E(X))^2)$ or the formula $E(X^2) - E(X)^2$. Recall that $E(X^2)$ can be expressed as $\sum_k k^2 \cdot P(X = k)$.

4. The Central Limit Theorem. Flip a fair coin $2n$ times and let X be the number of heads you get. In 1733, Abraham de Moivre observed that for **large** n and for **small** k, the probability $P(X = n + k)$ is approximately

$$\frac{1}{\sqrt{\pi n}} e^{-k^2/n}.$$

Use this to estimate the probability of getting heads between 2490 and 2510 times in 5000 flips of a fair coin. [Hint: Integrate.]