Math 309 **Fall 2014**
Homework 5 **Drew Armstrong**

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1. De Morgan's Law. Let U be a set and consider the following logical statement depending on an integer n. We will call this statement $P(n)$:

"For **any** n sets $A_1, A_2, \ldots, A_n \subseteq U$ we have $(A_1 \cup A_2 \cup \cdots A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$."

(a) Explain why $P(2)$ is a true statement.
(b) Fix $n \geq 2$ and **assume** for induction that $P(n)$ is a true statement. In this hypothetical case, show that the statement $P(n+1)$ is also true. [Hint: We proved something very similar in class.]
(c) What do you conclude?

Proof. The statement $P(2)$ says that "for **any** sets A_1 and A_2 we have $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$". Is this true? Yes. In fact this is just the statement of De Morgan's Law in the Boolean algebra of sets. We proved previously that De Morgan's Law holds in **any** Boolean algebra. [If you don't remember this, you can use Venn diagrams to prove it in this case.]

Now we fix some $n \geq 2$ and **assume for induction** that the statement $P(n)$ is true. That is, we assume that "for **any** sets $A_1, A_2, \ldots, A_n$ we have $(A_1 \cup \cdots \cup A_n)^c = A_1^c \cap \cdots \cap A_n^c$". **In this hypothetical case**, we want to show that the statement $P(n+1)$ is **also** true. That is, we want to show that "for **any** sets $A_1, A_2, \ldots, A_{n+1}$ we have $(A_1 \cup \cdots \cup A_{n+1})^c = A_1^c \cap \cdots \cap A_{n+1}^c$". How can we show this?

To show this, we consider any $n+1$ sets. Let's call them $A_1, A_2, \ldots, A_{n+1}$. [I hope you don't get confused that we keep calling the sets A_i. They really could be **any** sets.] The trick is to observe that

$$A_1 \cup \cdots \cup A_{n+1} = (A_1 \cup \cdots \cup A_n) \cup A_{n+1}.$$

Then the fact that $P(2)$ is true means that

$$(A_1 \cup \cdots \cup A_{n+1})^c = (A_1 \cup \cdots \cup A_n)^c \cap A_{n+1}^c.$$

Finally, since we **assumed** that $P(n)$ is true, we have

$$\begin{aligned}(A_1 \cup \cdots \cup A_{n+1})^c &= (A_1 \cup \cdots \cup A_n)^c \cap A_{n+1}^c \\ &= (A_1^c \cap \cdots \cap A_n^c) \cap A_{n+1}^c \\ &= A_1^c \cap \cdots \cap A_{n+1}^c.\end{aligned}$$

Hence $P(n+1)$ is true, as desired.

In summary, we have show that $P(2)$ is true, and **if** $P(n)$ is true **then** $P(n+1)$ is true. By the principle of **induction**, we conclude that $P(n)$ **is** true for all $n \geq 2$. □

2. Two Biased Dice. Suppose you have two 4-sided dice, one red and one blue. Suppose that each of these dice has probability distribution:

k	1	2	3	4
$P(k)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$

You roll the two dice and record the outcome.

(a) What is the sample space of this experiment?
(b) Compute the probability of each possible outcome. [Hint: Multiply.] Verify that the sum of the probabilities equals 1.

(c) What is the probability that "the sum of the dice is 6"?
(d) What is the probability that "the sum of the dice is 6 **or** the red die shows 3"?

(a) The sample space is the set of all sequences of length 2 with entries from the set $\{1, 2, 3, 4\}$. In other words,

$$S = \{1, 2, 3, 4\}^2.$$

If I want, I can write all the elements in a table:

	1	2	3	4
1	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$
2	$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$
3	$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$
4	$(4,1)$	$(4,2)$	$(4,3)$	$(4,4)$

(b) If the two dice were fair then each outcome would have probability $\frac{1}{\#S} = \frac{1}{16}$, but the dice are **not** fair. Note that the probabiliy of the outcome (i, j) is the probability that "the red die shows i **and** the blue die shows j". Since (we assume) that the two dice are independent, this is just the product of probabilities:

$$P((i, j)) = P(\text{the red die shows } i) \cdot P(\text{the blue die shows } j).$$

We can fill in these probabilites in our table:

	1	2	3	4
1	$\frac{4}{64}$	$\frac{8}{64}$	$\frac{2}{64}$	$\frac{2}{64}$
2	$\frac{8}{64}$	$\frac{16}{64}$	$\frac{4}{64}$	$\frac{4}{64}$
3	$\frac{2}{64}$	$\frac{4}{64}$	$\frac{1}{64}$	$\frac{1}{64}$
4	$\frac{2}{64}$	$\frac{4}{64}$	$\frac{1}{64}$	$\frac{1}{64}$

You can add up these 16 numbers and see that they sum to 1. Or you can observe that these 16 numbers are exactly the terms in the expansion of $1 = 1^2 = (\frac{1}{4} + \frac{1}{2} + \frac{1}{8} + \frac{1}{8})^2$, so **of course** they sum to 1.

(c) Let E be the event "the sum of the dice is 6". This corresponds to the set $E = \{(2, 4), (3, 3), (4, 2)\}$, so the probability is

$$\begin{aligned} P(E) &= P((2,4)) + P((3,3)) + P((4,2)) \\ &= \frac{4}{64} + \frac{1}{64} + \frac{4}{64} \\ &= \frac{9}{64}. \end{aligned}$$

(d) Let F be the event "the red die shows 3". We are looking for the probability of $E \cup F$. There are two ways to compute this. First, using brute force. Note that $F = \{(3, 1), (3, 2), (3, 3), (3, 4)\}$ and hence

$$E \cup F = \{(2, 4), (3, 3), (4, 2), (3, 1), (3, 2), (3, 4)\},$$

so the probability is

$$\begin{aligned} P(E\cup F) &= P((2,4)) + P((3,3)) + P((4,2)) + P((3,1)) + P((3,2)) + P((3,4)) \\ &= \frac{4}{64} + \frac{1}{64} + \frac{4}{64} + \frac{2}{64} + \frac{4}{64} + \frac{1}{64} \\ &= \frac{16}{64}. \end{aligned}$$

Second, using Inclusion-Exclusion:

$$P(E\cup F) = P(E) + P(F) - P(E\cap F).$$

We already know that $P(E) = \frac{9}{64}$ and $P(F) = \frac{1}{8} = \frac{8}{64}$ (because the probability that the red die shows 3 is completely independent of what the blue die does). We also know that $E\cap F = \{(3,3)\}$ (these are the outcomes in which the dice sum to 6 **and** the red die shows 3), and hence $P(E\cap F) = P((3,3)) = \frac{1}{64}$. Finally, we conclude that

$$\begin{aligned} P(E\cup F) &= P(E) + P(F) - P(E\cap F) \\ &= \frac{9}{64} + \frac{8}{64} - \frac{1}{64} \\ &= \frac{16}{64}. \end{aligned}$$

I think using Inclusion-Exclusion was the easier way to go.

3. The Birthday Problem. Suppose there are n people in a room and you record all of their birthdays as a number between 1 and 365 (assume no one was born on February 29).

(a) What is the sample space? How many elements does it have?
(b) Show that the number of outcomes in which no two people have the same birthday is

$$365 \cdot 364 \cdot 363 \cdots (365 - n + 1).$$

(c) Now let's assume that each of the 365 days is equally likely to be someone's birthday. In this case, what is the probability that **no two people have the same birthday**?
(d) Following from part (c), what is the probability that **there exist two people in the room with the same birthday**? Use a computer to find the smallest n such that this probability is greater than 1/2.

(a) We walk into a room containing n people. For each person we find out their birthday and record it as a number between 1 and 365. The set of possible outcomes is a sequence of length n whose entries are elements of the set $\{1, 2, \ldots, 365\}$. Hence the sample space is

$$S = \{1, 2, \ldots, 365\}^n.$$

The number of elements in this set is $\#S = 365^n$. Because we assume that all birthdays are **equally likely** (which is not quite true, but don't worry about it), the probability of any event $E \subseteq S$ can be computed as

$$P(E) = \frac{\#E}{\#S} = \frac{\#E}{365^n}.$$

This will help us later.

(b) Now let E be the event that "no two people have the same birthday". We want to show that $\#E = 365 \cdot 364 \cdot 363 \cdots (365 - n + 1)$. Note that E is just the set of sequences of length n in which no two entries are the same. We can create such a sequence starting at the beginning: There are 365 choices for the first entry. Then there are 364 choices for the second entry (because one has been used already). Then there are 363 choices for the third entry (because two have been used already). Since there are n entries we get

$$\begin{aligned}\#E &= (365-0)(365-1)(365-2)\cdots(365-(n-1)) \\ &= 365\cdot 364\cdot 363\cdots(365-n+1).\end{aligned}$$

(c) By the above discussion, the probability that no two people have the same birthday is

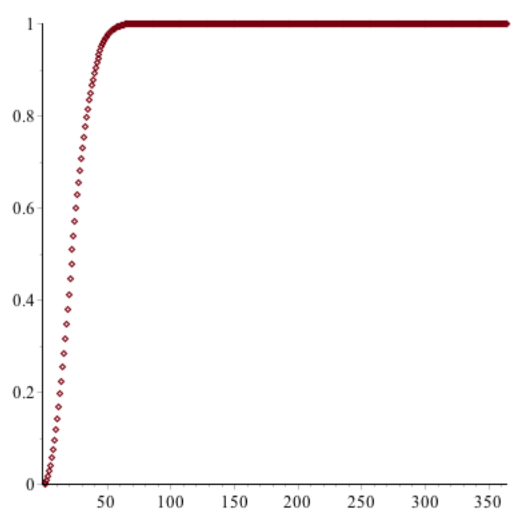
$$P(E) = \frac{\#E}{\#S} = \frac{365\cdot 364\cdot 363\cdots(365-n+1)}{365^n}.$$

(d) Note that the complement of $E =$"no two people have the same birthday" is $E^c =$"there exist two people with the same birthday". Thus the probability that there exist two people with the same birthday is

$$P(E^c) = 1 - P(E) = 1 - \frac{365\cdot 364\cdot 363\cdots(365-n+1)}{365^n}.$$

Note that this probability is a function of n. Let's call it $f(n)$. It starts with $f(1) = 0$ and $f(2) = \frac{1}{365}$. (Do these two values make sense to you?) And it ends with $f(n) = 1$ for any number of people $n \geq 366$. (Does that make sense to you?) As n goes from 1 to 366, the value of $f(n)$ increases from 0 to 1. At some point it must cross 1/2. When does this happen?

I can't do this by hand! I need a computer. My computer spits out the following graph of $f(n)$ for $n = 1, 2, \ldots, 366$.



It seems that the graph crosses 1/2 around $n = 25$. Looking at specific values we find that

$$\begin{aligned}f(21) &= 0.443 \\ f(22) &= 0.476 \\ f(23) &= 0.507 \\ f(24) &= 0.538\end{aligned}$$

Thus the smallest n for which $f(n) > \frac{1}{2}$ is $n = 23$. In a room full of 23 people, it is more likely than not that two of them share the same birthday. In our class of 40 students, the probabiliy that two students share the same birthday is $f(40) = 0.891$, or 89.1%. What is the smallest n for which $f(n)$ is greater than 99%?

4. Yahtzee. Suppose you roll 5 **fair** 6-sided dice.

(a) What is the sample space?
(b) Since the dice are fair, each outcome has the same probability. What is this probability?
(c) When all 6 dice show the same number, this is called "rolling a Yahtzee". What is the probability of rolling a Yahtzee? [Hint: How many ways can it happen?]
(d) Suppose you roll the dice n times. What is the probability that you will roll a Yahtzee **exactly k times**? [Hint: Each roll is equivalent to a biased coin flip.]
(e) Suppose you roll the dice 1000 times. What is the probability that you will roll a Yahtzee **at least once**?

(a) You roll 5 dice. Each die can take a value in the set $\{1,2,3,4,5,6\}$. We record the five values as a sequence. So our sample space is the set of sequences

$$S = \{1,2,3,4,5,6\}^5.$$

(b) Note that the sample space has size $\#S = 6^5 = 7776$. Since all outcomes are **equally likely** (the dice are assumed to be fair), the probabiliy of any particular outcome is $\frac{1}{7776}$. More genrally, the probability of any event $E \subseteq S$ is

$$P(E) = \frac{\#E}{\#S} = \frac{\#E}{7776}.$$

(c) Let E be the event "we roll a Yahtzee". Explicitly, we have

$$E = \{(1,1,1,1,1),(2,2,2,2,2),(3,3,3,3,3),(4,4,4,4,4),(5,5,5,5,5),(6,6,6,6,6)\}.$$

Thus the probability of rolling a Yahtzee is

$$P(E) = \frac{\#E}{\#S} = \frac{6}{6^4} = \frac{1}{6^4} = \frac{1}{1296}.$$

That's pretty rare.

(d) Now we will think of one roll of 5 dice as a "biased coin flip" with Yahtzee = heads and $P(\text{heads}) = \frac{1}{1296}$. If you flip a bised coin n times, the probability that you get heads (i.e., Yahtzee) exactly k times is

$$\binom{n}{k} P(\text{heads})^k P(\text{tails})^{n-k} = \binom{n}{k}\left(\frac{1}{1296}\right)^k \left(\frac{1295}{1296}\right)^{n-k}.$$

(e) Now suppose we roll the 5 dice 1000 times. What is the probability that we get Yahtzee **at least once**? It's much easier to consider the complementary event, i.e., that we get Yahtzee **exactly zero times**. By part (d) this probability is

$$\binom{1000}{0}\left(\frac{1}{1296}\right)^0 \left(\frac{1295}{1296}\right)^{1000} = 1 \cdot 1 \cdot \left(\frac{1295}{1296}\right)^{1000} = \left(\frac{1295}{1296}\right)^{1000}.$$

Hence the probability that we get a Yahtzee **at least once** is

$$1 - \left(\frac{1295}{1296}\right)^{1000} = 0.538 = 53.8\%.$$

Even though Yahtzees are rare, you're more likely than not to get one if you roll the dice 1000 times. How many times would you have to roll the dice to have a 99% chance of getting a Yahtzee?