Math 309 **Fall 2014**
Homework 2 Solutions **Drew Armstrong**

If S is a **finite** set, we let $\#S$ denote its number of elements. We call this the **size** or the **cardinality** of S. Sometimes we use the equivalent notation $|S| := \#S$.

1. Let X and Y be **finite** sets and let $f : X \to Y$ be a function.

(a) We say that $f : X \to Y$ is an **injection** if for all $y \in Y$ there is at *most* one $x \in X$ such that $f(x) = y$. If $f : X \to Y$ is an injection, show that $\#X \leq \#Y$.
(b) We say that $f : X \to Y$ is a **surjection** if for all $y \in Y$ there is at *least* one $x \in X$ such that $f(x) = y$. If $f : X \to Y$ is a surjection, show that $\#X \geq \#Y$.
(c) We say that $f : X \to Y$ is a **bijection** if it is both an injection and a surjection. If $f : X \to Y$ is a bijection, show that $\#X = \#Y$.

[Hint: For each $y \in Y$ let $d(y)$ denote the number of $x \in X$ such that $f(x) = y$. What can you say about the sum $\sum_{y\in Y} d(y)$?]

Before we begin, let $f : X \to Y$ be any function and for each $y \in Y$ let $d(y)$ be the number of elements $x \in X$ such that $f(x) = y$. Thus, $d(y)$ is the number of arrows pointing at y. If we sum the numbers $d(y)$ for all $y \in Y$ we are just counting all of the arrows. Since (by definition) there is **exactly one arrow** pointing from each element of X we conclude that

$$\#X = \sum_{y\in Y} d(y).$$

For part (a) we assume that $f : X \to Y$ is injective. In this case we have $d(y) \leq 1$ for all $y \in Y$. Summing over these numbers gives

$$\#X = \sum_{y\in Y} d(y) \leq \sum_{y\in Y} 1 = \#Y.$$

For part (b) we assume that $f : X \to Y$ is surjective. In this case we have $d(y) \geq 1$ for all $y \in Y$. Summing over these numbers gives

$$\#X = \sum_{y\in Y} d(y) \geq \sum_{y\in Y} 1 = \#Y.$$

For part (c) we assume that $f : X \to Y$ is bijective (i.e., both injective and surjective). Since f is injective we know from part (a) that $\#X \leq \#Y$ and since f is surjective we know from part (b) that $\#X \geq \#Y$. Putting these together, we conclude that $\#X = \#Y$.

2. If X and Y are finite sets, explain why there are $\#Y^{\#X}$ different functions from X to Y.

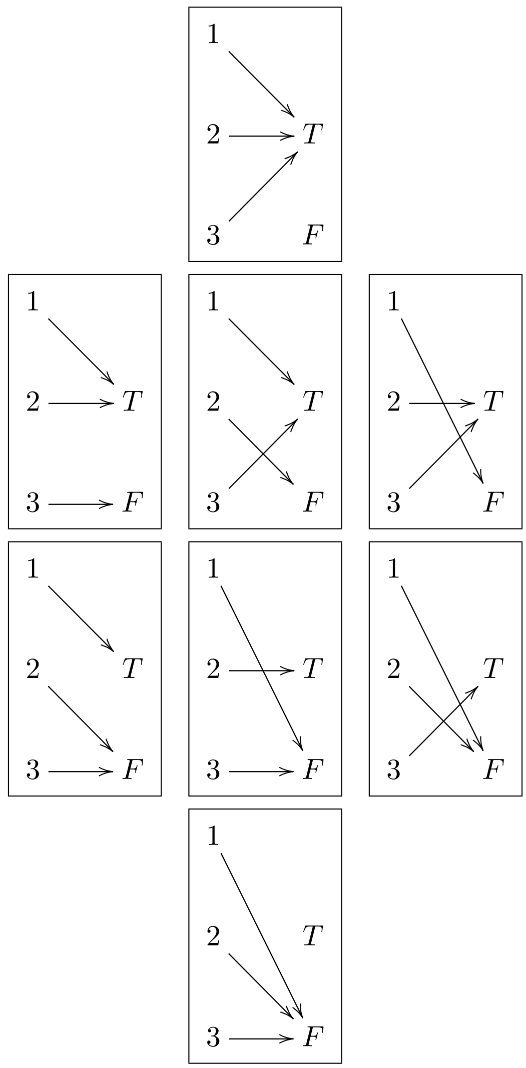
Recall that a **function** $f : X \to Y$ is a set of arrows of the form $x \to y$ (with $x \in X$ and $y \in Y$) with the property that for each $x \in X$ there is a **unique** $y \in Y$ such that $x \to y$. We call this unique element $y = f(x)$. Thus to specify a function $f : X \to Y$ we just need to choose the value $f(x)$ for each element $x \in X$. For each $x \in X$ there are $\#Y$ choices for $y = f(x) \in Y$. These choices can be made completely independently, so the total number of possibilities is

$$\underbrace{\#Y \times \#Y \times \cdots \times \#Y}_{\#X \text{ times}} = \#Y^{\#X}.$$

We conclude that the number of different functions from X to Y is $\#Y^{\#X}$.

3. Explicitly write down all of the functions from $\{1,2,3\}$ to $\{T,F\}$. How many are there? (See Problem 2.) How many of these functions are injective, surjective, bijective?

Here they are:



Note that there are $8 = 2^3 = \#\{T,F\}^{\#\{1,2,3\}}$ different functions, which agrees with the result of Problem 2. Note that 6 of the functions are surjective, 0 are injective, and 0 are bijective. In fact, we can use Problem 1 to see that no function from $\{1,2,3\}$ to $\{T,F\}$ can possibly be injective. Suppose, hypothetically, that we **did** have an injective function $f : \{1,2,3\} \to \{T,F\}$. Then Problem 1(a) implies that $\#\{1,2,3\} \leq \#\{T,F\}$, which is a contradiction.

4. Explicitly write down all of the subsets of $\{1,2,3\}$. Compare to your answer to Problem 3. Can you describe a bijection (one-to-one correspondence) between the set of functions $\{1,2,3\} \to \{T,F\}$ and the set of subsets of $\{1,2,3\}$?

Here they are:

$$\{1,2,3\}$$

$$\{1,2\} \quad \{1,3\} \quad \{2,3\}$$

$$\{1\} \quad \{2\} \quad \{3\}$$

$$\emptyset$$

I have arranged them to make clear that there is a one-to-one correspondence (i.e., a bijection) between the set of **functions** $\{1,2,3\} \to \{T,F\}$ and the set of **subsets** of $\{1,2,3\}$. We can describe this explicitly as follows.

Given a function $f : \{1,2,3\} \to \{T,F\}$ we define the subset

$$\Phi(f) := \{x \in \{1,2,3\} : f(x) = T\} \subseteq \{1,2,3\}.$$

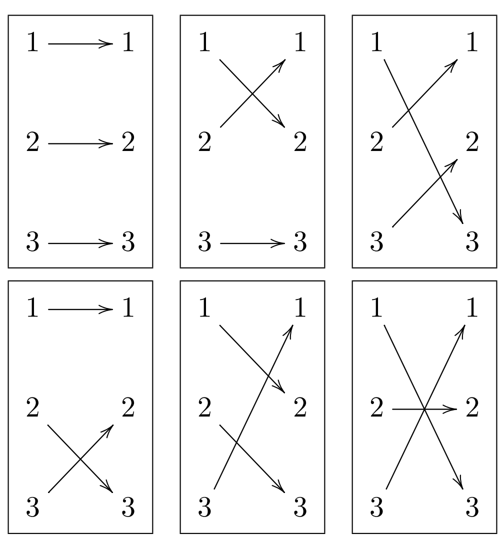
To verify that Φ is a one-to-one correspondence, we simply observe that it can be inverted. Given any subset $S \subseteq \{1,2,3\}$, note that $\Phi^{-1}(S)$ is the function $\{1,2,3\} \to \{T,F\}$ that sends x to T when $x \in S$ and sends x to F when $x \notin S$.

Maybe we could come up with some fancy notation for this but we won't bother. The main thing I want to point out is that because there exists bijection, Problem 1(c) implies that there are the same number of subsets of $\{1,2,3\}$ as there are functions $\{1,2,3\} \to \{T,F\}$: namely, $8 = 2^3$.

[Thinking Problem: Let X be any set with n elements. How many subsets does X have? Why?]

5. How many functions are there from $\{1,2,3\}$ to $\{1,2,3\}$? (Don't write them all down.) How many of the functions $\{1,2,3\} \to \{1,2,3\}$ are **bijections**? Explicitly write them down.

From Problem 2 we know that the number of functions $\{1,2,3\} \to \{1,2,3\}$ is $3^3 = 27$. I won't write them all down. Of these 27 functions, 6 are bijections. Here they are:



[Thinking Problem: Let X be any set with n elements. How many bijections $X \to X$ are there?]