Math 309 **Fall 2014**
Exam 1 (Wed Sept 24) **Drew Armstrong**

There are 5 problems, each with 3 parts. Each part is worth 2 points, for a total of 30 points. If any two exams are submitted with identical answers then **both** exams will receive 0 points.

1. Induction. For all positive integers n and p we define

$$S_p(n) := 1^p + 2^p + 3^p + \cdots + n^p.$$

In this problem you will use induction to prove that $S_1(n) = n(n+1)/2$ for all $n \geq 1$.

(a) Check that the formula $S_1(n) = n(n+1)/2$ is true for $n = 1$, 2, and 3.

n	$S_1(n)$	$n(n+1)/2$	correct?
1	1	$\frac{1(1+1)}{2} = \frac{1\cdot 2}{2} = 1$	✓
2	$1+2=3$	$\frac{2(2+1)}{2} = \frac{2\cdot 3}{2} = 3$	✓
1	$1+2+3=6$	$\frac{3(3+1)}{2} = \frac{3\cdot 4}{2} = 6$	✓

(b) Find a simple equation relating $S_1(n)$ and $S_1(n+1)$.

$$S_1(n+1) = 1 + 2 + 3 + \cdots + n + (n+1)$$
$$S_1(n+1) = S_1(n) + (n+1).$$

(c) Prove that **if** $S_1(n) = n(n+1)/2$, **then** $S_1(n+1) = (n+1)(n+2)/2$. You must begin your proof with the words "Consider some $n \geq 1$ and assume that…".

Consider some $n \geq 1$ and assume that $S_1(n) = n(n+1)/2$. In this case we have

$$\begin{aligned} S_1(n+1) &= S_1(n) + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= (n+1)\left(\frac{n}{2} + 1\right) \\ &= \frac{(n+1)(n+2)}{2}, \end{aligned}$$

as desired.

2. Recurrence. Consider the following recurrence relation:

$$\boxed{p_n = p_{n-1} + n.}$$

(a) **Assume** that $p_0 = 3$. In this case, make a table of p_n for n between 0 and 6.

n	0	1	2	3	4	5	6
p_n	3	4	6	9	13	18	24

(b) **Assume** that $p_5 = 20$. In this case, tell me the value of p_2.

We can rearrange the recurrence to get $p_{n-1} = p_n - n$. Then we have

$$\begin{aligned} p_4 &= p_5 - 5 = 20 - 5 = 15 \\ p_3 &= p_4 - 4 = 15 - 4 = 11 \\ p_2 &= p_3 - 3 = 11 - 3 = 8. \end{aligned}$$

(c) **Assume** that $p_0 = 1$. In this case, tell me a **closed formula** for p_n. [Hint: Prob 1.]

We have

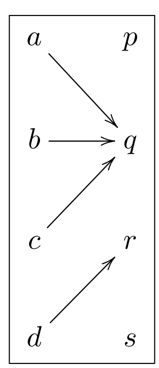
$$\begin{aligned} p_0 &= 1 \\ p_1 &= p_0 + 1 = 1 + 1 \\ p_2 &= p_1 + 2 = 1 + 1 + 2 \\ p_3 &= p_2 + 3 = 1 + 1 + 2 + 3 \\ &\vdots \\ p_n &= 1 + (1 + 2 + 3 + \cdots + n). \end{aligned}$$

Then using the formula from Problem 1 gives $p_n = 1 + \frac{n(n+1)}{2}$.

3. Functions. Consider the sets $X = \{a, b, c, d\}$ and $Y = \{p, q, r, s\}$.

(a) Draw an example of a function $f : X \to Y$ that is **not** injective and **not** surjective.

There are 232 different correct answers. Here is one of them:

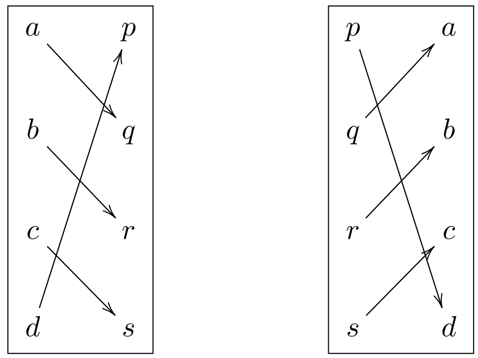


(b) Tell me the total number of different functions from X to Y.

In general the number of different functions from X to Y is $\#Y^{\#X}$. In this case we have $\#X = \#Y = 4$, hence there are $4^4 = 256$ functions.

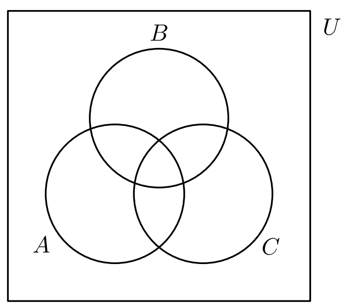
(c) Draw an example of an **invertible** function $f : X \to Y$. Also draw its inverse f^{-1}.

There are 24 different correct answers. Here is one of them:

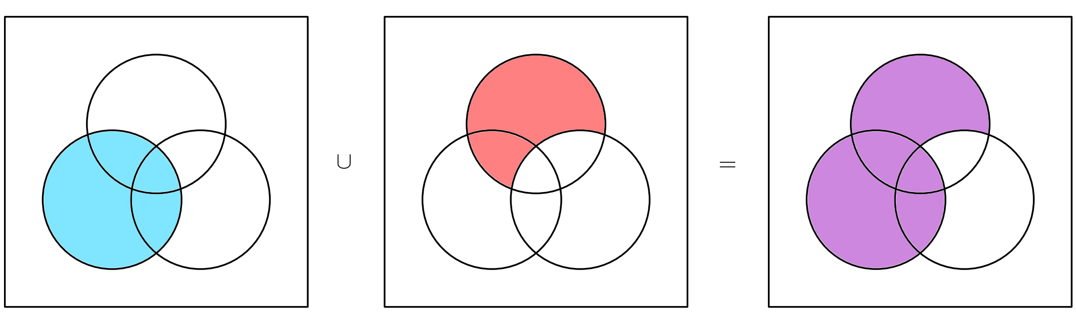


4. Venn Diagrams. Fix a universal set U and consider sets $A, B, C \subseteq U$. In this problem you will use Venn diagrams to prove that $A \cup (B \cap C^c) = (A \cup B) \cap (A \cup C^c)$.

(a) Draw a general Venn diagram displaying the sets A, B, C, and U.

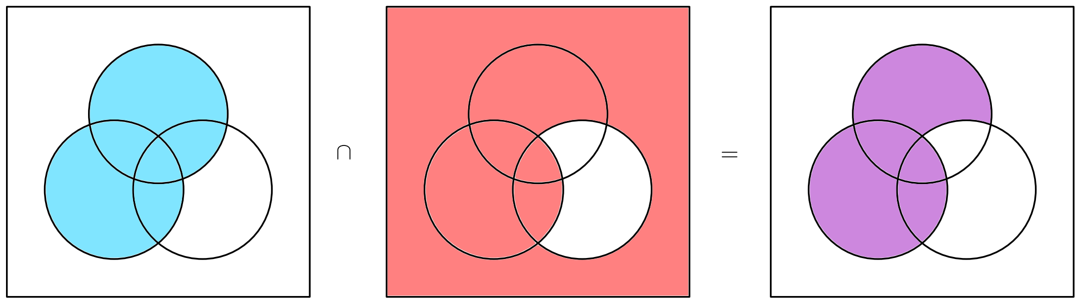


(b) Use Venn diagrams to draw the set $A \cup (B \cap C^c)$. Show intermediate steps.



A $\cup$ $B \cap C^c$ $=$ $A \cup (B \cap C^c)$

(c) Use Venn diagrams to draw the set $(A \cup B) \cap (A \cup C^c)$. Show intermediate steps.



$A \cup B$ $\qquad$ $A \cup C^c$ $\qquad$ $(A \cup B) \cap (A \cup C^c)$

[Remark: The fact that $A \cup (B \cap C^c) = (A \cup B) \cap (A \cup C^c)$ is an example of the distributive law.]

5. Truth Tables. Let P and Q be logical statements.

(a) Draw the truth table for the statement $P \wedge Q$ (i.e., P AND Q).

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

(b) Draw the truth table for the statement $\neg(P \wedge Q)$ (i.e., NOT (P AND Q))

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

(c) Draw the truth table for the statement $\neg P \vee \neg Q$ (i.e., (NOT P) OR (NOT Q)). [Hint: It may help to think about the Venn diagram of the set $A^c \cup B^c$.]

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

[Remark: The fact that $\neg(P \wedge Q) = \neg P \vee \neg Q$ is an example of de Morgan's law.]