What is a Number?

Today: The definition of $\mathbb{Z}$.

[see handout]

In many ways, Euclid's axioms have been replaced in modern mathematics by the axioms for natural numbers

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

and integers

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

We've been using properties of these sets implicitly for a while.

Now I'll show you the formal definition so we can be more confident in our proofs. These axioms will serve you in all of your future math courses.

Definition: Let $\mathbb{Z}$ be a set equipped with four concepts

"$=$", "$\leq$", "$+$", "$\times$".

These concepts satisfy approximately 20 axioms (depending on how you count). Luckily, all of these axioms are "obvious" statements that you are already comfortable with.

We discussed the axioms of "$=$" and "$\leq$" last time. These are roughly equivalent to Euclid's common notion (but we're being more careful than he was).

Now let's take a look at the axioms of addition:

$\downarrow$

(A1) $\forall a, b \in \mathbb{Z}$, $a+b = b+a$

(A2) $\forall a, b, c \in \mathbb{Z}$, $a+(b+c) = (a+b)+c$

(A3) $\exists 0 \in \mathbb{Z}$, $\forall a \in \mathbb{Z}$, $a+0 = a$.

(A4) $\forall a \in \mathbb{Z}$, $\exists b \in \mathbb{Z}$, $a+b = 0$

In modern jargon these axioms say that $\mathbb{Z}$ is a group under addition. The element $0 \in \mathbb{Z}$ is called the identity element of the group.

Prop: The identity element is unique.

Proof: Suppose we have two elements $z_1, z_2 \in \mathbb{Z}$ satisfying

① $\forall a \in \mathbb{Z}$, $a + z_1 = a$

② $\forall a \in \mathbb{Z}$, $a + z_2 = a$.

Then we conclude that

$$z_1 \overset{②}{=} z_1 + z_2 \overset{①}{=} z_2$$ //

Since the identity element is unique we can give it a special name. We call it "0".

The element b in (A4) is called an additive inverse of a.

Prop: Additive inverses are unique.

Proof: Let $a \in \mathbb{Z}$. Suppose we have two elements $b_1, b_2 \in \mathbb{Z}$ satisfying

(1) $a + b_1 = 0$
(2) $a + b_2 = 0$.

Then we conclude that

$$\begin{aligned} b_1 &= b_1 + 0 && \text{(A3)} \\ &= b_1 + (a + b_2) && \text{(2)} \\ &= (b_1 + a) + b_2 && \text{(A2)} \\ &= (a + b_1) + b_2 && \text{(A1)} \\ &= 0 + b_2 && \text{(1)} \\ &= b_2 + 0 && \text{(A1)} \\ &= b_2 && \text{(A3)} \end{aligned}$$

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Since the additive inverse of a is unique we can give it a special name. We will call it "$-a$".

Additive inverses now allow us to define a new operation called subtraction.

Definition: For all $a, b \in \mathbb{Z}$ we define

$$"a - b" = a + (-b).$$

Now let's discuss the axioms of multiplication. We will write the product of a & b as ab.

(M1) $\forall\, a, b \in \mathbb{Z}$, $ab = ba$.
(M2) $\forall\, a, b, c \in \mathbb{Z}$, $a(bc) = (ab)c$.
(M3) $\exists\, 1 \in \mathbb{Z}$, $\forall\, a \in \mathbb{Z}$, $1a = a$.

You can check that the element 1 is unique. We call it the multiplicative identity element of $\mathbb{Z}$. [We call 0 the additive identity element.].

Note that there is no axiom (M4) because integers do not necessarily have multiplicative inverses.

Example: Define the integer $2 = 1+1$. There does not exist $q \in \mathbb{Z}$ such that $2q = 1$.

[The proof of this requires the well-Ordering Axiom, which we haven't discussed yet.]

We also require an axiom telling us how addition and multiplication interact.

(D) $\forall a, b, c \in \mathbb{Z}$, $a(b+c) = ab + ac$.

Q: How does the additive identity 0 interact with multiplication?

A: $\forall a \in \mathbb{Z}$, $0a = 0$

To prove this we will use a Lemma

* Cancellation Lemma:

For all $a, b, c \in \mathbb{Z}$ we have

$$(a + b = a + c) \Rightarrow (b = c).$$

Proof:

$$\begin{aligned} b &= b + 0 && \text{(A3)} \\ &= b + (a + (-a)) && \text{(A4)} \\ &= (b + a) + (-a) && \text{(A2)} \\ &= (c + a) + (-a) && \text{assumption} \\ &= c + (a + (-a)) && \text{(A2)} \\ &= c + 0 && \text{(A4)} \\ &= c. && \text{(A3)} \end{aligned}$$

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[We use (A1) so often that it gets too tedious to mention it.]

Prop: $\forall a \in \mathbb{Z}$, $0a = 0$.

Proof: Let $a \in \mathbb{Z}$. Then we have

$$\begin{aligned} 0 + 0 &= 0 && \text{(A3)} \\ (0+0)a &= 0a \\ 0a + 0a &= 0a && \text{(D)} \\ \cancel{0a} + 0a &= \cancel{0a} + 0 && \text{(A3)} \\ 0a &= 0 && \text{cancellation} \end{aligned}$$

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Q: Why don't we take $0a = 0$ as an axiom?

A: Because we don't need to!

On HW3 you will investigate how subtraction interacts with multiplication.

You will show that $\forall a, b \in \mathbb{Z}$ we have

- $-(-a) = a$
- $(-a)b = a(-b) = -(ab)$
- $(-a)(-b) = ab$.

Have you ever wondered why

negative × negative = positive ?

The reason is because it follows logically from the obvious properties of addition and multiplication.

We have no choice!

Properties of Order and The Well-Ordering Axiom

Last time we discussed the axioms

(A1) — (A4), (M1) — (M3), (D).

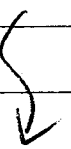
In modern jargon, these 8 axioms tell us that $(\mathbb{Z}, +, \times, 0, 1, =)$ is a ring.

But recall that $\mathbb{Z}$ also has a strict total order "$<$" satisfying

- $\forall a, b, c \in \mathbb{Z}$, $(a < b \wedge b < c) \Rightarrow (a < c)$
- $\forall a, b, c \in \mathbb{Z}$, exactly one of the following statements is true:

$$a < b \quad , \quad a = b \quad , \quad b < a$$

We need 3 more axioms to tell us how "$<$" interacts with the other ring structures:



Axioms of Order:

(O1) $\forall a,b,c \in \mathbb{Z}$, $(a<b) \Rightarrow (a+c < b+c)$.

(O2) $\forall a,b,c \in \mathbb{Z}$, $(a<b \wedge 0<c) \Rightarrow (ac < bc)$

(O3) $0<1$.

These 11 axioms now tell us that

$$(\mathbb{Z}, +, \times, 0, 1, =, <)$$

is an ordered ring. But this can't be the full definition because there exist other ordered rings, such as

$\mathbb{Q}$ & $\mathbb{R}$

fractions real numbers.

Q: What is special about $\mathbb{Z}$ that distinguishes it among all of the ordered rings?

[Remark: Let us define the notation "$a \leq b$" = "$a<b$ or $a=b$".]

This leads to the least obvious and most important axiom of the integers.

✡ Well-Ordering Axiom.

Let $S \subseteq \mathbb{Z}$ be a set of integers and assume that

- S is not empty ($S \neq \emptyset$)
- S has a lower bound. ($\exists b \in \mathbb{Z}, \forall s \in S, b \leq s$).

Then the set S has a least element, i.e.

$\exists m \in S, \forall s \in S, m \leq s.$ //

This axiom is complicated and it will take some time to learn how to use it. Here is a first example.

Theorem: There are no integers between 0 and 1.

↓

Proof: Consider the set of positive integers,

$$S = \{ n \in \mathbb{Z} : 0 < n \}.$$

This set is non-empty (because $1 \in S$) and it is bounded below (by 0). Hence by Well-Ordering there exists a smallest positive integer $m \in S$

I claim that $m = 1$.

Indeed, since $1 \in S$ and since m is the least element of S we must have

$$m \leq 1$$

Now assume for contradiction that

$$m < 1$$

Then multiplying $0 < m$ & $m < 1$ by m gives $0 < m^2$ & $m^2 < m$, which contradicts the minimality of m.

Q.E.D.

This axiom is logically much more complicated than the others and it took a long time to realize its importance. [It was first stated by Giuseppe Peano in 1889 in an equivalent form called the "principle of induction".]

Application of Well-Ordering

Here's our first application.

Theorem: Let $\alpha \in \mathbb{R}$ and $\alpha \notin \mathbb{Z}$. Then there exists an integer $m \in \mathbb{Z}$ such that

$$m-1 < \alpha < m.$$

Proof: Define the set

$$S := \{ n \in \mathbb{Z} : \alpha < n \} \subseteq \mathbb{Z}.$$

Since this set is non-empty and bounded below, it has a least element; call it $m \in S$. By definition we have

$$\alpha < m.$$

Now consider $m-1 \in \mathbb{Z}$. Since $m-1 < m$ and since m is the least element of S we conclude that $m-1 \notin S$, i.e.,

$$\alpha \not< m-1.$$

In other words, $m-1 \leq \alpha$. Finally, since $\alpha \notin \mathbb{Z}$ and $m-1 \in \mathbb{Z}$ we know that $m-1 \neq \alpha$, hence

$$m-1 < \alpha. \qquad ///$$

[You will use this on HW3 to prove that for all $d \in \mathbb{Z}$ we have

$$\sqrt{d} \notin \mathbb{Z} \Rightarrow \sqrt{d} \notin \mathbb{Q}. \quad]$$

From this point on (unless otherwise stated) we will use the axioms of $\mathbb{R}$ rather informally. Instead of taking every proof all the way back to the axioms, we will take it to the point where we are confident that we could take it back to the axioms if we really had to (but we never will really have to).

This is how formalism is usually treated in mathematics. It's like insurance; it's there if we need it, and we hope we don't need it.

The Principle of Induction

Last time we finished discussing the definition of $\mathbb{Z}$, including the most important and least-obvious axiom.

★ Well-Ordering Axiom:

- Any non-empty subset of $\mathbb{Z}$ that is bounded below has a least element.
- Any non-empty subset of $\mathbb{Z}$ that is bounded above has a greatest element.

Here's a joke application.

Theorem: There are no uninteresting natural numbers.

Proof: Suppose for contradiction that there exists an uninteresting natural number and let $S \subseteq \mathbb{N}$ be the set of these.

Since $S \neq \emptyset$ (by assumption) and since S is bounded below (by 0), Well-Ordering implies that S has a smallest element, say $m \in S$.

But then m is "the smallest uninteresting natural number", which is interesting. This contradicts the fact that $m \in S$. //

Remark: By contrast, there are plenty of uninteresting real numbers because $\mathbb{R}$ does not satisfy Well-Ordering.

Here's a more serious application. Our original proof of $\sqrt{2}$ had some gaps because we never proved the following two statements:

- Every fraction can be written in lowest terms.
- Every integer is of the form $2k$ or $2k+1$, for some $k \in \mathbb{Z}$, but not both.

Now I'll give a fully rigorous proof using Well-Ordering:

Theorem: $\sqrt{2} \notin \mathbb{Q}$.

Proof: Suppose for contradiction that $\sqrt{2} \in \mathbb{Q}$, so we can write $\sqrt{2} = a/b$ for some $a, b \in \mathbb{Z}$ with $b \geq 1$.

Now define the set

$$S = \{ n \in \mathbb{N} : n \cdot \sqrt{2} \in \mathbb{Z} \} \subseteq \mathbb{N}.$$

Note that $S \neq \emptyset$ because $b \geq 1$ and $b\sqrt{2} = a \in \mathbb{Z}$ imply that $b \in S$. So by Well-Ordering there exists a smallest element $m \in S$.

Now we will try to find a contradiction. Since $\sqrt{2} \notin \mathbb{Z}$, we proved last time that there exists an integer $c \in \mathbb{Z}$ such that

$$c < \sqrt{2} < c+1$$
$$0 < \sqrt{2} - c < 1.$$

Multiply everything by m to get

$$0 < m(\sqrt{2} - c) < m.$$

If we can show that $m(\sqrt{2} - c) \in S$ then this will be a contradiction because m is the smallest element of S.

To show this, note that

$$m(\sqrt{2} - c) = \underbrace{m\sqrt{2}}_{\in \mathbb{Z}} - \underbrace{mc}_{\in \mathbb{Z}} \in \mathbb{Z}$$

and then $0 < m(\sqrt{2} - c) \implies m(\sqrt{2} - c) \in \mathbb{N}$.
Finally, note that

$$m(\sqrt{2} - c)\sqrt{2} = \underbrace{2m}_{\in \mathbb{Z}} - \underbrace{cm\sqrt{2}}_{\in \mathbb{Z}} \in \mathbb{Z}.$$

We conclude that $m(\sqrt{2} - c) \in S$, as desired.

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The nice thing about this proof is that it easily generalizes to prove the following.

Theorem: For all $a \in \mathbb{Z}$ we have

$$\sqrt{a} \notin \mathbb{Z} \implies \sqrt{a} \notin \mathbb{Q}.$$

Proof: Homework. //

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Discussion: We have already seen two equivalent statements of the Well-Ordering Axiom. There are many more, Maybe the most famous version is called the "Principle of Induction".

☆ Principle of Induction:

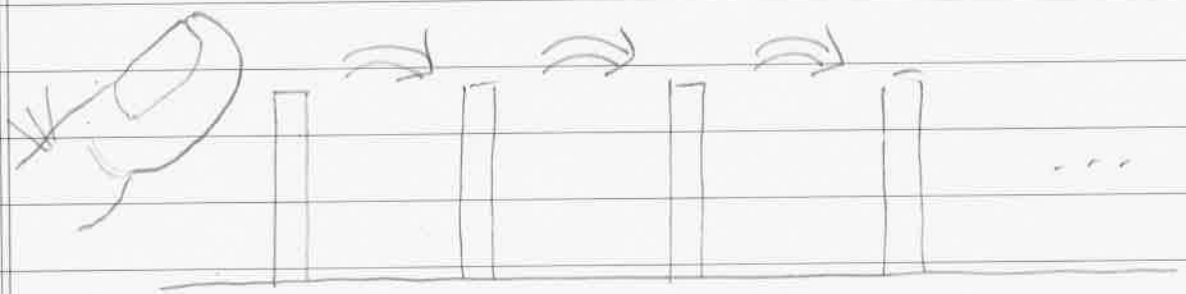
Let $P(n)$ be a statement about the integer $n \in \mathbb{Z}$. IF

(1) $P(b) = T$ for some $b \in \mathbb{Z}$, and
(2) $\forall n \in \mathbb{Z}_{\geq b}$, $P(n) \implies P(n+1)$,

then we conclude that $P(n) = T$ for all integers $n \geq b$. //

[I don't expect you to be able to absorb this the first time you see it. In my experience it takes students quite a while to absorb what this is saying. Don't worry; you will have lots of practice. I just wanted to put it out there today so your sub-conscious can start thinking about it.]

I think of induction as follows. We want to knock down a line of dominoes:



Step (1) is your finger and step (2) is gravity. These two contributions are very different and both of them are necessary if you want to knock down all of the dominoes.

Example: Let $n \in \mathbb{Z}$ and consider the statement $P(n) =$ "$n < 2^n$". We would like to prove that $P(n) = T$ for <u>all</u> $n \geq 0$.

HOW?

Let's check some small cases:

$P(0) =$ "$0 < 1$" $= T$ ✓
$P(1) =$ "$1 < 2$" $= T$ ✓
$P(2) =$ "$2 < 4$" $= T$ ✓
⋮

I could have my computer check many more cases, but eventually the computer and I will both be dead.

In order to prove that $P(n) = T$ for <u>all</u> (infinitely many) $n \geq 0$ we need some kind of abstract principle. This is exactly what induction does for us.

Here's the argument:

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Let n be some fixed but arbitrary integer greater than 1, and assume for induction that $n < 2^n$. In this case we have

$$n+1 < n+n < 2^n + 2^n = 2 \cdot 2^n = 2^{n+1}.$$

We have shown that for all $n \geq 2$ we have

$$P(n) \Longrightarrow P(n+1).$$

Since we already checked that $P(2) = "2 < 4" = T$, the Principle of Induction now tells us we are allowed to say that

$$P(n) = T \quad \forall \; n \geq 2.$$

Since we also checked that $P(0) = P(1) = T$, we can say that

$$P(n) = T \quad \forall \; n \geq 0.$$

Division With Remainder

We have now fully discussed the definition of $\mathbb{Z}$ and it's time to start developing the theory of $\mathbb{Z}$. This is the subject of

Number Theory.

We will systematically develop some of the main theorems in this subject and then we will discuss some applications [in particular, to cryptography].

Recall that $\mathbb{Z}$ is not a "group" under multiplication because integers do not (usually) have multiplicative inverses. Our first theorem will tell us how to recover some sort of "division" in $\mathbb{Z}$.

✡ The Division Theorem:

Given integers $a, b \in \mathbb{Z}$ with $b \neq 0$,

(1) $\exists\, q, r \in \mathbb{Z}$ such that

$$a = qb + r \quad \& \quad 0 \leq r < |b|.$$

(2) These q & r are unique. We call them "the" quotient and "the" remainder of a modulo b.

Proof: For part (1) consider any integers $a, b \in \mathbb{Z}$ with $b \neq 0$. It is easy to find $q, r \in \mathbb{Z}$ with $a = qb + r$, but can we do it such that $0 \leq r < |b|$?

Define the set

$$S = \{ a - nb : n \in \mathbb{Z} \}$$
$$= \{ \ldots, a-2b, a-b, a, a+b, a+2b, \ldots \}.$$

Since $b \neq 0$, this set must contain a non-negative number.

$\{$

Let $r \in S$ be the smallest non-negative number in S (which exists by well-ordering). By definition of S, there exists $q \in \mathbb{Z}$ such that $r = a - qb$, and hence $a = qb + r$. By definition we also have $0 \leq r$.

Now I claim that $r < |b|$. To prove this, assume for contradiction that $|b| \leq r$. Then we have

$$\begin{aligned} |b| &\leq r \\ |b| - |b| &\leq r - |b| \\ 0 &\leq r - |b|. \end{aligned}$$

On the other hand, since $b \neq 0$ we have $r - |b| < r$. Finally, since

$$r - |b| = a - qb - |b| = a - (q \pm 1)b$$

we conclude that $r - |b| \in S$. This contradicts the fact that r is the smallest non-negative element of S.

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We have shown that q & r exist with the desired properties. For part (2) we will show that they are unique.

To do this, suppose that we have $q_1, q_2, r_1, r_2 \in \mathbb{Z}$ such that

$$a = q_1 b + r_1 \quad \& \quad a = q_2 b + r_2$$
$$0 \leq r_1 < |b| \qquad\qquad 0 \leq r_2 < |b|.$$

In this case we will show that

$$q_1 = q_2 \quad \& \quad r_1 = r_2.$$

Suppose for contradiction that $r_1 \neq r_2$. Without loss of generality, let's say that $r_1 < r_2$. Then we have

(*) $$0 = r_1 - r_1 < r_2 - r_1 \leq r_2 < |b|.$$

Then since $q_1 b + r_1 = a = q_2 b + r_2$, we have

$$q_1 b + r_1 = q_2 b + r_2$$
$$q_1 b - q_2 b = r_2 - r_1$$
$$(q_1 - q_2) b = (r_2 - r_2)$$

Since $r_1 \neq r_2$, we have $r_2 - r_1 \neq 0$ so that from a property discussed in class,

$$|b| \leq |r_2 - r_1| = r_2 - r_1 ,$$

which contradicts (*). We conclude that $r_1 = r_2$ and hence

$$(q_1 - q_2) b = 0 .$$

Since $b \neq 0$, this implies [why?] that $q_1 - q_2 = 0$ and hence $q_1 = q_2$.

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We conclude that the quotient and remainder of a mod b (mod is short for "modulo") are unique.

QED.

[Remark: At the end of the proof we used the fact that $\mathbb{Z}$ satisfies

☆ Multiplicative Cancellation:

Given $a, b, c \in \mathbb{Z}$ with $a \neq 0$ we have

$$ab = ac \implies b = c.$$

You will prove this on HW3.]

The Division Theorem finally allows us to prove a fact that we've been taking for granted for a long time.

Definition: Let $n \in \mathbb{Z}$. We say that n is <u>even</u> if $\exists\, k \in \mathbb{Z}$ with $n = 2k$ and we say that n is <u>odd</u> if $\exists\, \ell \in \mathbb{Z}$ with $n = 2\ell + 1$.

↓

Theorem: Every integer is either even or odd; not both, not neither.

Proof: Consider any $n \in \mathbb{Z}$. Since $2 \neq 0$, the Division Theorem says that there exist unique $q, r \in \mathbb{Z}$ such that

$$n = q \cdot 2 + r \quad \& \quad 0 \leq r < 2$$

Since $0 \leq r < 2$ implies $r \in \{0, 1\}$ we see that n is either even or odd, and it can't be both because the remainder is unique. //

Here's another basic fact we can finally prove.

Theorem: There does <u>not</u> exist an integer $a \in \mathbb{Z}$ such that $2a = 1$.

Proof: Suppose for contradiction that there <u>does</u> exist $a \in \mathbb{Z}$ such that $2a = 1$. This implies that

$$1 = a \cdot 2 + 0,$$

so the remainder of 1 modulo 2 is 0. On the other hand, we have

$$1 = 0 \cdot 2 + 1,$$

so the remainder of 1 mod 2 is 1. Since $0 \neq 1$ this contradicts the uniqueness of the remainder.

Pretty good, huh? I think we've now proved all of the obvious properties of integers that we once assumed.