Math 230 **Spring 2019**
Homework 3 **Drew Armstrong**

Fun With Axioms

Problem 1. Let $a, b \in \mathbb{Z}$. Use the axioms of $\mathbb{Z}$ to prove the following properties:

(a) $-(-a) = a$.
(b) $a(-b) = (-a)b = -(ab)$. [Hint: Multiply both sides of $b - b = 0$ by a.]
(c) $(-a)(-b) = ab$. [Hint: Combine parts (a) and (b).]

Problem 2. Use the axioms of $\mathbb{Z}$ to prove the following properties:

(a) $\forall a \in \mathbb{Z}, (0 < a) \Leftrightarrow (-a < 0)$. [Hint: Add something to both sides.]
(b) $\forall a, b, c \in \mathbb{Z}, (a < b \wedge c < 0) \Rightarrow (bc < ac)$. [Hint: Use 2(a) and 1(b).]
(c) $\forall a, b \in \mathbb{Z}, (a \neq 0 \wedge b \neq 0) \Rightarrow (ab \neq 0)$. [Hint: There are 4 cases.]
(d) **Multiplicative Cancellation.** $\forall a, b, c \in \mathbb{Z}, (ab = ac \wedge a \neq 0) \Rightarrow (b = c)$. [Hint: If $ab = ac$ then $a(b - c) = 0$. Use the contrapositive of 2(c).]

Problem 3. For all $a \in \mathbb{Z}$ we assume that $\sqrt{a} \in \mathbb{R}$ exists. In this problem you will show that

$$\sqrt{a} \notin \mathbb{Z} \Rightarrow \sqrt{a} \notin \mathbb{Q}.$$

(a) Assume that $\sqrt{a} \notin \mathbb{Z}$. Prove that there exists $m \in \mathbb{Z}$ such that $m - 1 < \sqrt{a} < m$. [Hint: Let $S = \{n \in \mathbb{Z} : \sqrt{a} < n\}$ and use Well-Ordering.]
(b) Now assume for contradiction that $\sqrt{a} \in \mathbb{Q}$ and consider the set $T := \{n \geq 1 : n\sqrt{a} \in \mathbb{Z}\}$. Use Well-Ordering to show that this set has a least element $d \in T$. But then show that $d(\sqrt{a} - m + 1)$ is a smaller element of T. Contradiction.

Problem 4. Let $a, b, c \in \mathbb{Z}$. Prove the following properties of divisibility:

(a) If $a|b$ and $b|c$ then $a|c$.
(b) If $a|b$ and $a|c$ then for all $x, y \in \mathbb{Z}$ we have $a|(bx + cy)$.
(c) If $a|b$ and $b|a$ then $a = \pm b$. [Hint: Use 2(d).]